# Proceedings of the International Workshop on Mathematics Education for Non-Mathematics Students Developing Advanced Mathematical Literacy

国際研究集会「高水準の数学的リテラシーのための大学数学教育」報告集

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Proceedings of the International Workshop on
Mathematics Education for Non-Mathematics Students Developing Advanced Mathematical Literacy

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「高水準の数学的リテラシー」概念下の教育デザイン・実施・継続的改善とその理論
共催：工学院大学

# Preface

Currently, university mathematics education for non-mathematics students is gaining international attention. Various practical studies have been conducted over the past two decades: attention on quantitative literacy has been increasing in the U.S.; science, technology, engineering, and mathematics (STEM) education has become widespread, and there has been a lot of research focusing on mathematics education for students studying engineering, economics, and so on. In Japan, it is imperative to establish a rational framework for mathematics education for non-STEM students in particular, in order to foster mathematical competence as citizens, businesspersons, or specialists in some field. Although there have been several studies on mathematics education design thus far, more research efforts are required to ensure they are effective in the actual context of Japanese society.

The themes of this workshop were as follows:

I. Mathematics Education for Science, Technology, and Engineering Students

II. Mathematics Education for Non-STEM Students to Promote Quantitative Literacy

Along the above themes, the following topics were addressed at the workshop:

- Successful learning contexts while studying linear algebra and calculus
- Developing information and communications technology (ICT) tools fostering students' understanding of mathematical concepts
- Epistemological analysis of mathematical knowledge
- Acquiring quantitative mathematical literacy at the university level
- Developing mathematics courses that foster the ability to utilize mathematics in real-world situations

The themes and topics of this workshop were based on a discussion that took place during an international workshop on "Mathematical Literacy at the University Level and Secondary-Tertiary Transition"* that was held in 2014. Following this discussion, we decided to expand the scope of our workshop.

The workshop was attended by 73 participants including invited speakers. We had 5 invited talks, 6 oral presentations, and 12 poster presentations. At the workshop, we exchanged results of research and practices on university mathematics education both in Japan and in other countries. We hope that the discussions that took place during the workshop will contribute to further advancement in research and educational practices on university mathematics education.

[*] Mizumachi, R. (ed.) (2017). *Mathematics Literacy in University Education*. Toshindo. [In Japanese]

# はじめに

数学を専門としない一般の大学生への数学教育は世界的に関心を集めている極めて重要なテーマです [1][2]。これらのテーマに関しては二十年近く前から，アメリカにおける数量的リテラシーへの注目[3]，世界的な広がりを持つ STEM 教育の実践，あるいは工学部教育における Artigue の問題提起[4] など様々な実践や議論が行われてきました。また，日本国内の議論に目を向ければ，数学を現実世界で活用する能力の育成という観点から，とくに非理工系学生のための数学教育の拡充が日本の大学教育の課題として指摘されています[5][6]。これまで教育デザインの指針はいろいろな形で提案され，それぞれに高い有効性が期待できるものの，現場で直ちに役に立つものとなるにはさらなる研究が必要であると思われます。

本研究集会は，

Ⅰ　理工系のための大学数学教育

Ⅱ　学士課程目標の一つとしての数量的リテラシーの教育

の２つをテーマとして企画され，具体的なトピックとして以下の５つが設定されました。

- 線形代数や微積分の習得における効果的な学習のデザイン
- 概念理解を促進する ICT ツールの開発
- 数学的知識の認識論的分析
- 大学レベルで身につけるべき数量的リテラシー
- 現実世界で数学を活用する能力を育成する数学科目の開発

これらのテーマおよびトピックは，2014 年に開催された国際研究集会「Mathematical Literacy at the University Level and Secondary-Tertiary Transition[7]」での議論に基づいています。本研究集会は，2014 年の議論を基盤としつつ，さらなる研究の進展と教育実践の広がり，および国際交流に貢献することを目指して開催されました。

研究集会には７３名が出席し，招待講演５件（うち海外からの招待講演２件），口頭発表６件，ポスター発表１２件が行われました。本研究集会での議論が大学数学教育の研究と教育実践のさらなる進展に寄与するものとなることを願っています。

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# Program

## プログラム

## Program プログラム

### Jan. 7 (Sunday)　1 月 7 日（日）

*Main Venue*: A-0652 (6th floor of the high rise building)

09:30-10:30 Registration

10:30-10:45 Opening

10:45-11:30 Ryuichi Mizumachi (Shonan Institute of Technology)

Advanced mathematical literacy and designs of 1st year mathematical courses for STEM students

11:30-12:00 Hiroshi Komatsugawa (Chitose Institute of Science and Technology)

A model of flipped classroom using an adaptive learning system

12:00-12:30 Satoru Takagi (Kogakuin Univ.), Sei-ichi Yamaguchi, Ryuichi Mizumachi (Shonan Institute of Technology)

Designing 1st year calculus courses

(Lunch)

14:00-14:45 [Invited Talk]

Masaaki Ogasawara (Hokkaido Univ.)

Struggle for the next stage of higher learning with special reference to science and technology

14:45-15:15 Discussion

*Theme: mathematics education for the first year students in engineering/science courses*

(Coffee break)

15:30-16:15 [Invited Talk]

Masami Isoda (Univ. of Tsukuba)

Mathematization: A theory for mathematics curriculum design

(Coffee break)

16:30-17:30 [Invited Talk]

Carl Winsløw (Univ. of Copenhagen)

Task design for university mathematics education with a case from engineering

18:00-20:00 Banquet　Cafeteria (7th floor of the middle class building)

Jan. 8 (Monday)　　１月８日（月・祝）

*Main Venue*: A-0652 (6th floor of the high rise building)

*Poster Session Venue*: A-0656 (6th floor of the high rise building)

09:00-09:30 Registration

09:30-10:15 [Invited Talk]

Kazunori Yamaguchi (Rikkyo Univ.)

Statistical thinking, communication and leadership

10:20-11:20 Poster Session

11:30-12:30 [Invited Talk]

Marianna Bosch (Univ. Ramon Llull)

Study and research paths in university education: linking inquiry and content-based teaching

(Lunch)

14:00-14:45 Mitsuru Kawazoe (Osaka Pref. Univ.)

Designing mathematics education based on the classification of human activities

14:45-15:15 Hirofumi Ochiai (Nagoya Bunri Univ.)

The analysis and practice of mathematics education based on the concept of affordance

15:15-15:45 George Gotoh (Niigata Univ.)

Understanding mathematical structures through problem-solving

(Coffee Break)

15:50-16:20 Discussion

*Theme: Quantitative Literacy*

16:20-16:30 Closing

# Invited Talks

## 招待講演

# Task Design for University Mathematics Education with a case from Engineering

**Carl Winsløw**
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**Abstract**: We present some main notions of *task design* as a viewpoint and approach for the problems related to teaching mathematics at university, with a specific focus on the teaching of mathematics to students in engineering programmes. We also present some examples of task analysis from ongoing work in the context of a large first-year course in mathematics for engineering students.

**Keywords**: task design, mathematics, engineering

## 1. Introduction

Any teaching situation is a joint work of students and one or more teachers. The purpose of the situation is invariably that the students should learn something, and whether they do so depend crucially on the students' work in the situation. Even in large audience lectures, students must do something in order to learn (pay attention, reflect, perhaps take notes). But it is clear that in much current university mathematics teaching, it is the teacher who appears to be most active, and on whose initiative almost all activity depends. As university teachers, we are naturally tempted to focus on our own activity when we think about teaching. But thinking of *the students' work* as the main "motor" of student learning – and hence, the main avenue to achieve the aims of university teaching – should make us change our main focus to what we can do to *frame and optimize the students' work*, in view of what they should learn.

In mathematics, we often think of students' work in terms of bigger or (often!) smaller tasks which they should be able to solve. In addition to attending lectures, students are usually asked to solve exercises of various kinds, both during the teaching and when the time comes to assess whether they have learned what they should. These assignments (exercises, projects etc.) should be a main focus of university mathematics teachers, for at least two reasons: (1) within a usual university course, they are the most direct and specific request for students to do mathematical work, and so the most direct way that we have to shape this work; (2) if students believe that the assignments are well aligned with the requirements involved in summative assessment (final exams etc.), then they are likely to invest serious efforts in doing them. In this paper, we will provide some examples and directions for how university mathematics teachers may pursue such a focus, with an emphasis on the teaching of students of engineering and other fields, for whom mathematics is not the main subject.

## 2. Task design in mathematics education

In mathematics education research, the area of *task design* [1] studies the properties and effects of tasks given to students. We can consider, broadly speaking, "properties" as everything we can say about the task from its formulation and our knowledge or assumptions about the students which are to solve it. By contrast, studying "effects" require observations or other forms of empirical data related to students actually working with the task. Teachers usually do not perform systematic studies of the tasks they give to students, but nevertheless engage informally with their properties and effects: when constructing or selecting tasks for their students, and when relating to students' work with tasks in different settings of their courses, including that of summative assessment (tasks set for exams and other contexts where the aim is to measure, rather than to evaluate, student knowledge). We now briefly outline what teachers can learn from task design research.

Concerning the properties or qualities of tasks, a crucial notion from task design theory is that of *didactic variable* [1,2]. For example, consider the following two tasks (see [3] for more on the mathematics and didactics involved with this particular example):

1. Show that every increasing function $f: [0,1] \rightarrow [0,1]$ has at least one fixed point.
2. Does every increasing function $f: [0,1] \rightarrow [0,1]$ have a fixed point?

The main different appears to be the language form: the first task is formulated as an order and the second as a question. In practice, the essential mathematical challenge may perhaps not differ much; but there is in fact more to do in order to achieve a correct answer to 2., since *the solution is not given*. Whether or not the solution is given, is an example of a generic didactic variable. It is meaningful in a variety of mathematical contexts and for a wide range of mathematical tasks. Another didactic variable is the use of specialized terminology in the formulation of the task (here, *increasing function* and *fixed point*) which could be adapted to supposed prerequisites of students. Didactic variables for tasks are, in short, any property which a task may have and which may be determined objectively by looking at the task formulation; moreover, the term "didactic" indicate an expectation or experience that the property will have some effect on students' learning from doing the task. In Sec. 4, we consider in more detail a set of didactic variables used to analyze mathematical project assignments in an engineering course, and reflect upon further design.

Concerning effects, teachers can develop an informal, experimental practice based on didactic variables. The main point is *the explicit attention to a specific property of the task,* such as the above (solution being given or not); such a property is of course not chosen randomly, but is based on a hypothesis that it may be important for realizing certain learning goals (in the example, such goals could for instance be that students reason about functions using a variety of representations, cf. [4]). Then, the teacher may observe the effects of varying this particular point – for instance, when giving version 1. and 2. of the tasks above to two halves of a reasonably homogeneous students population, in a context where students' communication and inquiry is emphasized. Developing an experimental approach to task design, based on systematic and explicit use of didactic variables, is an important way for teachers to turn their attention to (and to learn more about) the framing of students' work.

## 3. Task design for students of non-mathematics majors

Most mathematics teaching at university level is given to students who attend university with other professions and subjects as their main goal – mostly through dedicated *service courses* (like "mathematics for biology", "linear algebra in finance " etc.). It is well known that mathematics courses in such programmes often meet with a range of problems, including

- Difficulty for students in the transition from secondary to university mathematics (a general problem [5] which is sometimes particularly acute for non-mathematics majors)
- Students' lack of motivation for and effort in mathematics classes[6]
- High rate of student failure at exams, in some cases affecting the whole programme as a "bottle neck" problem [6]
- Even for students who pass their university mathematics courses, other courses may report that they lack the expected mathematical knowledge and skills, and these are not always

identical to what is taught in the mathematics courses (a problem highly dependent on the major subject)

Of course, an increased focus on task design cannot solve *all* such problems. But there are a few general strategies which are suggested by large parts of the literature. Most concern the necessity of analyzing the *connections* between the contents and requirements of the mathematics course on the one hand, and the concerned students' backgrounds and foregrounds on the other. Their *background* is previous learning of mathematics and in particular the skills and knowledge they are actually capable of mobilizing in new contexts (for instance, in tasks whose appearance is different from what they encountered before). Their *foreground*, the rest of the study programme and subsequent professional tasks, and in particular those parts where mathematical knowledge is needed.

A major problem is that university mathematics teachers may be simply not know enough about either background or foreground: they may have outdated or unrealistic conceptions of what students "should" know and be able to do from secondary school, and depending on their own academic profile (often in pure or applied mathematics) they may also lack insight into what students will need subsequently. Regarding the first point, *diagnostic testing* (to identify shortcomings in mathematical skills and concepts among students) can be a useful tool [7]. In many countries *bridging courses* have been established, in which both diagnosis and remedial of such shortcomings is attempted (e.g. [8]). To enhance the connectivity between mathematics courses and other parts of the study programme, a collaboration between university mathematics teachers and teachers of other modules can be a strategy to develop new tasks and other activities *within* the mathematics course (see Section 4 for a concrete example). In general, such tasks are often developed within some vision of *mathematical modelling*, where the tasks involve more or less rich questions from the major discipline, while drawing on mathematical contents from the mathematics course. To construct meaningful and manageable tasks of this kind, which in particular connect also to the mathematics course, is in general a highly non-trivial endeavor.

It should also be noted that task design may not suffice to establish satisfactory alignment with students' backgrounds and foregrounds. Sometimes the syllabi and other specifications of the contents and aims of mathematics courses need also to be reviewed, although this will typically go much beyond what individual teachers can do. But again, it will certainly require contributions from university teachers whose combined expertise covers both mathematics and the major.

## 4. Case: task design in Engineering Mathematics

The Technical University of Denmark (DTU) is one of the largest and most prestigious schools of engineering in Europe (no. 11 according to the 2017 ranking of www.topuniversities.com), enrolling well over 2000 undergraduate students every year. About 1100 of them, spread on 17 different B.Sc.Eng. programmes, have a common and mandatory mathematics course (referred to as Mat1) in their first year, where it takes up 1/3 of their study time. It covers a sequence of mathematics topics which can be found in similar courses at "classical" engineering programmes around the world: complex numbers and linear algebra up to diagonalization and eigenvalues, calculus in one and several variables including systems of differential equations and vector calculus up to the divergence theorem. Throughout the course, specific uses of the computer algebra software *Maple* (cf. https://www.maplesoft.com/) are demonstrated by teachers and elaborated by students, and the course homepage (https://01005.compute.dtu.dk/) is used to publish notes, Maple sheets and videotaped lectures, and other materials.

The common curriculum for all 17 programmes (from Architectural Engineering to Strategic Analysis and System Design) makes the "connection" problem somewhat difficult, as the mathematical needs certainly vary among these. According to the main responsible of Mat1, the course is also considered an “identifying common element” of the B.Sc.Eng.-programmes, beyond the specific needs in each of the programmes.

Nevertheless, Mat1 does not ignore the connectivity problem. Besides homework exercises presenting standard “applications” of the materials, as well as more demanding “thematic assignments” [9] done throughout the year, the course has (since the year 2000) an important element called the Mat1 “project” [10, which is a main source for this section]. It is the design and profile of these projects which we will now outline.

A Mat1 project is an assignment containing about 20-30 more or less challenging tasks, all related to a specific “engineering problem” such as “how to design a heating system for a house with one or more rooms”. The students work with this project in groups of about 6 people from the same study programme; several project assignments are proposed, with a choice of 4-5 assignments for each study programme. Here, some assignments may be considered relevant for more than one study programme – the total number of assignments proposed in a given year is around 15. Some assignments are revised and reused in subsequent years, based on a thorough evaluation of the outcomes; a total of 36 assignments have been used (in different versions) over the past 10 years. After the groups have compiled their “project report” (responding to the assignment), they defend it at an oral examination, and the grade obtained make up for 25% of the total grade in the course. These incentives suffice to make the students work very hard with the assignment and to deliver quite extensive and often creative reports, typically including extensive documentation of *Maple* use. There is also massive evidence that the students find the assignments both engaging and challenging.

The format and profile of the assignments created over the past 17 years vary considerably. Naturally, it is a considerable and delicate work to construct a project assignment of this type. A couple of overall characteristics of the process and products of this work should be emphasized:

- The *origin* of a project assignment can either come from a specific problem or piece of research in a science or engineering discipline, or from (applied) mathematics itself. This also roughly corresponds to personal initiative: for instance, a colleague from a specific department or research group proposes an outline of a project related to his specific field, sometimes in the hope that students who work on it will later choose to specialize in this field. It also happens that a Mat1-teacher hear of a relevant problem and invite colleagues to help creating a corresponding Mat1-project.
- The engineering part is usually presented rather extensively in the assignment, including “mathematical models” and data. The main tasks of the assignment consist in (challenging) uses of the models, data and Mat1 techniques – including *Maple*-use – to solve a major engineering problem.
- The *drafting* and subsequent *revisions* of the assignment is invariably carried out by one or more Mat1-teachers, in order to optimize the alignment of the assignment with contents and methods of Mat1, according to principles which have recently been made more explicit, as didactic variables (see [10] and below).

At the end of the oral exam, all examining teachers and external referees gather to share experiences and suggestions for the use in subsequent use (or non-use) of the years’ assignments.

A typical example of a Mat1 project assignment is *Heat flow in a house – simulation and dimensioning*. Two settings are considered subsequently: first, a house with one room, where the model is a first order linear ODE, and then a house with three rooms (or “zones”), with a variety of external conditions including sunshine, varying temperature and so on; for this latter part, the students draw on many parts of Mat1, including semi-advanced linear algebra and complex functions, as well as *Maple*-based computational techniques. The project assignment is based on an original source in the form of a research paper published by a colleague from DTU. It touches upon several branches of engineering: besides the obvious connection to building and energy engineering, there is a surprising link between electrical circuits and the model of energy flow in the house which is used in the project. The students also work to find optimal insulation solutions, based on a simple model for investment, together with real data from the construction industry and for energy prices.

As is often the case, the first Mat1-project assignments were created by a small group of enthusiastic teachers, based on intuition and personal network. Over the years, the process has become more systematic, and together with the course manager, the author recently compiled a set of 10 didactic variables which sum up the guiding principles which have formed (more or less implicitly) over the years. They are as follows:

DV1. *Mathematical breadth and depth*. A Mat1-project may let students use several areas of Mat1-content in combinations, and at theoretical and technical level well beyond the "standard tasks" which occurred in their first meeting with the contents.

DV2. *New mathematical contents*. A Mat1-project can let students discover or work with mathematical contents that extends the Mat1-syllabus.

DV3. *Maple use.* From standard use to more advanced uses which the students have not encountered in Mat1 (these are then normally explained).

DV4. *Source of problem(s).* From advanced textbook problem to original and recent research paper.

DV5. *Engineering breadth.* From very specialized area to involving more branches of engineering.

DV6. *Modelling.* The mathematical model is often simply given, but students could have to work more or less independently with its structure and details.

DV7. *Realism.* The model could more or less simplified, so less or more realistic.

DV8. *Data.* The data could be authentic and used as in the source, or more or less simplified.

DV9. *Information search.* Students might have to search for information (data, terms, methods), but in most cases not beyond given resources from the course.

DV10. *Solution.* The assignment may enable students to find a more or less "complete" (or "satisfying") solution to the overall problem.

With these didactic variables we have analyzed all the 36 assignments from the past 10 years [10]. Using a simple coding, the analysis is really quite simple and relatively objective when viewed relatively to the whole inventory of assignments.

In subsequent work, we plan to investigate their potential for leading to a more systematic task design in this context, for instance identifying and realizing missing potentials in specific assignments, investigating dependencies among variables (in both assignments and student reports) and their cause, etc. We have presented the above didactical variables here because we think they could be helpful (as a whole or in part) for practitioners in similar courses and institutions who wish to design "advanced tasks for undergraduate engineering mathematics", as a strategy for overcoming at least some of the problems outlined at the beginning of Section 3.

## 5. Conclusion and perspectives

Systematic and adaptive work with task design is likely to become a still more significant element in research based development of university mathematics education, and also in the skills that university mathematics teachers will have to master. There are several reasons for this, such as:

- The use of digital technologies open new possibilities not only for distance teaching (including the distribution of tasks and solution assessment based on variables), but also for students mathematical work based on CAS and similar tools.

- Students' mathematical backgrounds and foregrounds change more rapidly than in the past, and include a still wider variation among students who study some mathematics at university;
- In part due to the proliferation of digital tools, most of these students will *not* need to do routine mathematical tasks by hand in their professional life, but will need to use mathematics in more creative and sometimes also more theoretical ways – thus, "inquiry" oriented mathematical performance are likely to gain more weight, both in formative and summative tasks for students, and such tasks are often more demanding to construct.

To close, I would like to make an observation, in view of the last point above. It is somewhat paradoxical that inquiry oriented mathematics teaching is still relatively rare at many universities around the world, but appear to be quite widespread in primary school in certain countries. Many researchers in the West have noted *Japanese [primary school] teachers' ability to design and implement high-quality mathematics lessons that are centered on high-quality mathematical tasks* [1, p. 34], and cite "lesson study" (*jyugyo-kenkyu*) and other forms of teacher collaboration as one reason for this success. Maybe there is a lesson to learn for us from primary school, concerning the value of collaborative design and dissemination of resources among mathematics teachers? In other words: could task design at university level benefit from more collaboration and resource sharing within and across universities – based on explicit declaration of didactical variables of the designs?

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# Study and research paths in university education: linking inquiry with content-based teaching

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**Abstract**: The Anthropological Theory of the Didactic proposes a new teaching format, study and research paths (SRP), based on the approach of open questions and including the study of knowledge contents. SRPs are not only a tool to design and implement new instructional processes; they can also be used to describe and analyze traditional teaching, and to question its main assumptions. During the past few years, our research group in Barcelona has implemented various types of SRPs in Engineering, Management and Chemistry degrees. They provide rich evidence of the world of possibilities they can open up when the main institutional constraints hindering teaching innovation in our universities are taken into account.

**Keywords**: anthropological theory of the didactic, inquiry, study and research paths, didactic ecology, mathematics as a service subject.

## 1. From "visiting works" to "questioning the world"

Within the Anthropological Theory of the Didactic, the pedagogical paradigm that prevails in university education – and also at other educational levels – has been characterized as the *paradigm of visiting works* [1]. Students are shown a predetermined set of bodies of knowledge – or knowledge "works" – that are considered important for them to learn. A team of teachers is assigned to help students study these works, that is, to help students get in contact with them, know their main features and use them in a specific way and under specific circumstances. The study process can then be described as a "visit" students carry out under the guidance of the teachers, who arrange the set of works in the best way they can in order to make the visit as pleasant, interesting and productive as possible. In this paradigm, the importance of the works to visit is taken for granted and it is mainly the teachers' responsibility to find the best possible conditions to introduce them, illustrate their main uses and purposes. To do so, teachers rely on several resources – textbooks, treatises, websites, selected papers, encyclopedias, etc. – elaborated by scholars and other teachers through a complex process of didactic transposition [2] under which bodies of knowledge are selected, organized, structured, sequenced; collections of problems are proposed to illustrate their main use; and names are given to all these constructions: Calculus, Analysis, Linear Algebra, Abstract Algebra, Ordinary Differential Equations, Graph Theory, etc.

The paradigm of visiting works can be related to a number of didactic phenomena in university education. One of them is the uniformity and slow evolution of the mathematical organizations that are proposed to be taught. In many countries, it is not unusual to find similar syllabi for a first course of mathematics in different degrees, such as Economics, Biology or Chemistry. What might change are the "applications", that is, the problems that are proposed at the end of each theme to show the utility of the notions introduced in the domain considered (economy, biology, chemistry, etc.). This stability can be explained by the logic underlying the sequencing of mathematical content, usually based on the theoretical organization of notions, properties and theorems that precedes putting them into use. We use the term "applicationism" [3] to describe this teaching epistemology, where the works of knowledge are first introduced in a predefined sequence and then "applied" to a variety of possible situations. Here, the didactic transposition process seems to reverse the historical order of knowledge construction. In fact, it tends to eliminate the questions that have motivated the construction of the knowledge works, in order to present a nicely structured organization of the answers given to these questions. All happens as if the questions were not part of the pieces of knowledge to be disseminated, but just a way to show their utility.

When we put a set of *questions* to study – and to answer – at the centre of the study process instead of a set of selected mathematical works that students have to visit, we obtain the *paradigm of questioning the world* [1]. In this case, the works of knowledge do not appear as monuments to visit, but as tools or instruments to provide answers to the questions under study, and also to generate more questioning about the considered answers and the corresponding realities. From this perspective, the paradigm of visiting works can be seen as the study of answers to questions that have been lost. The following words of a Spanish educator ([4], our translation) very well illustrates the weaknesses of the situation thus created:

> It makes no sense to give answers to those who have never asked the question; therefore, the basic task of the teacher is to recover the questions, concerns, the process of looking for the men and women who developed the knowledge now listed in our books. [...] We need to give up on the professions of faith in the organized answers of our books. We should make our students look towards the world around and rescue the initial questions, making them think.

The main element to define the paradigm of questioning the world is the notion of *study and research path* (SRP) based on the so-called *Herbartian schema*:

$$[S(X; Y; Q) \rightarrow M] \hookrightarrow A^{\heartsuit},$$

where $S(X; Y; Q)$ represents a didactic system formed by a group of students $X$, a group of teachers $Y$ and a question $Q$ to be studied. The study of $Q$ generates an *inquiry* process that leads to the elaboration of a final answer $A^{\heartsuit}$. The inquiry process involves different types of objects or resources that form a *didactic milieu* $M$:

$$M = \{A_1^{\Diamond}, A_2^{\Diamond}, \ldots, A_m^{\Diamond}, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p, D_{p+1}, D_{p+2}, \ldots, D_q\}.$$

The $A_i^{\Diamond}$ are "ready-made" *answers* to $Q$ (or to some questions $Q'$ derived from $Q$) that the students $X$, supervised by $Y$, have found and related to $Q$. These answers $A_i^{\Diamond}$ are "hallmarked" by an institution that presents them as the "official" answers to the considered questions – hence the "stamp" $\Diamond$. The $W_j$ are *works* drawn upon to make sense of the $A_i^{\Diamond}$, analyse and "deconstruct" them, and to build up $A^{\heartsuit}$. The $Q_k$ are the questions induced by the study of $Q$ and of the $A_i^{\Diamond}$, as well as the questions raised by the construction of $A^{\heartsuit}$. Finally, the $D_l$ are sets of data of all natures gathered in the course of the inquiry.

The paradigm of questioning the world can be considered as an enlargement of the paradigm of visiting works in the following sense: during the inquiry, the group of students $X$ not only needs to find appropriate labelled answers $A^{\Diamond}$ that are supposedly productive for the inquiry, they also need to use them in a suitable way to elaborate the final answer $A^{\heartsuit}$. To do this, it is sometimes necessary to explore large domains of knowledge and the help of expert guides may be required. Answers $A^{\Diamond}$ have to be *studied*, and thus "visited", but with a clear aim: elaborate an answer $A^{\heartsuit}$ to $Q$. There is, however, a critical difference between both paradigms: the "visit" in the last case is always motivated by the supposed productivity of $A^{\Diamond}$ in the construction of $A^{\heartsuit}$, not by the importance of $A^{\Diamond}$ itself. Moreover, the inquirers $X$ and $Y$ are allowed to discard any work of knowledge $A^{\Diamond}$ that does not appear useful for the inquiry, independently of its epistemological prestige. This is very different from the situation where inquiry activities are proposed as a means to learn a given pre-established piece of knowledge, tool or competence. In the words of Yves Chevallard ([1], p. 183):

> In too many cases, the so-called "inquiry-based" teaching resorts to some form or another of "fake inquiries", most often because the generating question $Q$ of such an inquiry is but a naive trick to get students to meet and study works $O$ that the teacher will have determined in advance. Of course, this is the plain consequence of the domination of the paradigm of visiting works, which implies that curriculum contents are defined in terms of works $O$. In contradistinction, in the paradigm of questioning the world, the curriculum is defined in terms of *questions* $Q$. However, the works $O$ studied in consequence of inquiring into these questions $Q$ play a central role in the process of defining and refining the curriculum: starting from a set of "primary" questions, the curriculum contents eventually studied will include the questions $Q$ and answers $A$, together with [other available answers] $A$ and the works $O$.

## 2. The methodology of study and research paths

During this past decade, our research group has designed and implemented various *study and research paths* (SRP) at university level in different degrees: Chemistry ([3]), Management ([5], [6]), Engineering ([7]), Health Sciences ([8]) and Teacher Education ([9]). They have been organized considering the specific condition of each university setting, in terms of number of students and teachers, length and number of sessions, time schedule, available facilities, etc. However, they all follow a similar format that can be summarized in the following points:

(1) At the starting point of the process, students are asked to act as a consultancy team, the teacher (or teachers) assuming the role of the leader and the students of junior consultants. The teacher presents a question $Q$ that is supposed to come from an imaginary client to whom an answer in the form of a report has to be handed in after a given period of time (some weeks or months). During the SRP, some interactions with the client are possible (for instance requesting more information by e-mail), and some intermediate reports are required.

(2) During the SRP, students are organised in small teams and different responsibilities are assigned to each team, according to the derived questions $Q_i$' generated by the SRP. Results of all teams are regularly shared, for instance through the presentation of partial reports and their discussion in the large group. A record of the discussions' outcomes and the decisions made is usually kept by the teacher or by a student acting as a "secretary". The work is then resumed to approach new derived questions till the elaboration of the final answer $A^{\heartsuit}$. It is important to note that the teacher does not know this final answer and that the initial question $Q$ is open enough not to accept only one specific answer.

(3) In the last SRPs experimented, a specific work was performed with the elements of the Herbartian schema that appear during the inquiry process, especially the derived questions $Q_i'$ raised, the external answers $A_j^{\Diamond}$ found and the partial answers $A_k^{\heartsuit}$ elaborated by the class. This specific work consists in describing the process followed as a question-answer map (QA map) that keeps record of the paths followed and helps plan the new foreseen ways (figure 1). It helps elaborate a narrative of the inquiry carried out, organise the results obtained and the work to be done. It also supports students in the unusual responsibility of raising questions, sharing and discussing them, as well as deciding – and reporting – on the paths to explore and those to discard.

(4) The SRP finishes when a final answer $A^{\heartsuit}$ is produced and considered ready to be submitted to the "client". The assessment strategies proposed all include the intermediate oral or written reports, as well as the final presentation of the answer given to $Q$ or to the derived questions $Q'$ assigned to each team. Panels with external teachers and experts are organised at times,

including oral or poster presentations.

A critical issue is the choice of the initial question $Q$ that leads the inquiry. Where does it come from? In all the experimented cases, the SRPs were implemented in traditional university settings fully immersed in the paradigm of visiting works. In particular, syllabi were always defined in terms of works to visit and competences to acquire, not in terms of questions to study. The first step of the SRP methodology supported by the paradigm of questioning the world consists precisely in *questioning* the selection of works that define the subject to be taught, trying to find, for each piece of knowledge proposed to be studied, some possible questions $Q_i$ that could give it some raison d'être, some specific utility. This utility has to be distinguished from the "applications" that are just illustrations of possible uses, but not real questions appearing in the professional domains related to the university degree. For instance, in the Management degree, the study of a human resource problem of workers' relocation or the logistic problem raised by the organisation of a bike-renting in a city can be modelled with transition matrices $M$ formed by the percentage of workers/bikes moving from one head office/parking lot to another. This leads to the consideration of $M^n$ as the distribution of workers/bikes after $n$ periods of time, the study of which requires a variate set of linear algebra tools, from elementary matrix calculation to diagonalization. In the same domain, the forecast of the number of users of a given social network (or of the sales of a given shop) leads to the mobilisation of functional tools and to connecting them with linear regression (or with epidemic models expressed in terms of differential equations), all of them important tools in a first course of Calculus ([5], [6]). In these cases, and due to the curriculum constraints, even if the initial question is open and there is no pre-established answer to it, the teacher can guide the students through the paths that are considered, if not more important, at least closest to the content of the subject.

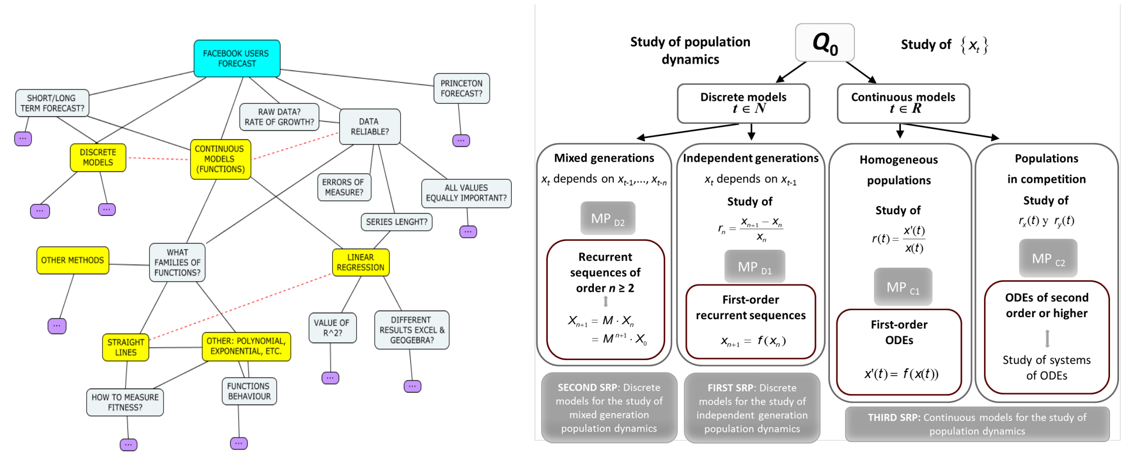


*Figure 1. Examples of question-answer maps: How to forecast Facebook users (left) and Population dynamics (right)*

In the case of Health Science, the work carried out by Catarina Lucas ([8]) proposes an alternative structuring and organisation of elementary differential Calculus based on the consideration of discrete and continuous models to forecast a disease outbreak. In this case, continuous models based on the derivative appear as a way to simplify discrete models based on the rate of growth, thus providing a radical change in the traditional raison d'être assigned to derivatives. Another interesting case in this respect is the SRP experimented some years ago by Barquero ([3]), which started from a question related to the evolution of populations and led to a long inquiry process engaging almost all the mathematical tools of a first year course for Chemistry, Biology or Geology (Figure 1).

## 3. The problem of the ecology of study and research paths

The experimentation of SRPs in traditional university degrees provides important information about their *ecology*, that is, the set of conditions and constraints that enable and hinder the implementation of this type of inquiry processes as normalized instructional formats. It is obvious that the usual way of teaching at university should be transformed to include SRPs. The modalities experimented were diverse: from a workshop running parallel to the lectures and problem solving sessions, to a full integration of the SRP in the normal teaching during the whole course or only for a few weeks.

An "ideal" organization would be an SRP leading the course during a period of time – depending on the generating power of the initial question approached – and some traditional lectures and problem solving sessions giving support to the SRP when some new works of knowledge $A_j^{\lozenge}$ are considered supposedly useful for the inquiry. The same SRP teacher or other experts can then be called to help the group of students incorporate the new knowledge into their milieu $M$. Once a given SRP is finished, another one can follow, till the end of the course in question.

An important constraint to take into account is the disciplinary organization of knowledge at the university. An initial open question $Q$ does not necessarily belong to a given knowledge domain (mathematics, physics, biology, etc.) but can usually be related to many of them. For instance, the case of a disease breakout is a problem of public health, but also a political issue; it can be located in the field of biology, but it can also be approached using mathematical models based on empirical data coming from health science, etc. So the disciplinary delimitation of knowledge at the university when it concerns its dissemination, appears as an important constraint to the implementation of the SRP. In contrast, the existence of cross-disciplinary subjects in some degrees – for instance as a preparation for the final degree project – offers more interesting conditions for this kind of inquiry processes.

The individualistic conception of learning is another important cultural constraint that hinders the organization of inquiry processes led by teams of students-investigators who will need to share responsibilities and contribute to the collective elaboration of a final answer. Unfortunately, what seems to be clear in a collective activity like a sports team, where different players assume different roles and are not supposed to act in the same way, is not always acceptable when dealing with knowledge affairs – even if, afterwards, team-work is one of the competences employers value the most... In the case of mathematics, this constraint is even harder since pedagogical tradition for teamwork is lacking in this area: students are used to solving problems on their own and getting feed-back for their individual work. The image of small teams of students solving problems at the wall in different corners of the classroom is rarely observed in ordinary university mathematical lessons.

An important constraint any teacher who leads an SRP feels, is related to the traditional *didactic contract* established between the teachers and the students, that is, the way responsibilities are currently shared in the mathematical activities carried out during the instructional process. At the university, as well as at secondary level, teachers are supposed to be experts in the domain they teach, preparing students to become "little experts" in a part of the domain. Teachers are supposed to be able to answer students' questions, to provide feed-back on their work, to give them an overview of the domain and to guide them through its different components. However, in the case of an inquiry process – and very much like in a PhD supervision –, teachers do not know the answer to the approached questions beforehand and cannot be experts in all the possible domains that might contribute to elaborating the final answer $A^{\heartsuit}$. It is not their responsibility to provide all the elements of the milieu $M$ (empirical data, derived questions, labelled works, etc.), or to provide all the intermediate answers that mark the process till the end. Contrary to what they are used to do, in

SRPs teachers have to learn how to help students plan the inquiry process, raise question, discuss their appropriateness, find empirical data, validate their quality or reliability, deconstruct the available answers, defend the final one, etc. [10] Of course, providing new information through traditional formats like lectures and problem solving sessions does not disappear, but, to the teachers' despair, tends to occupy a secondary position…

All in all, what finally appears as the most difficult obstacle to overcome is the new relationship teachers have to maintain with mathematical knowledge – or their discipline background. One of the most important deficits regarding the design and implementation of SRPs is related to the kind of knowledge resources needed by teachers to organize their management. In the case of traditional teaching, thank to centuries of didactic transposition work – like Euclid's Elements, Euler's treatises, Cauchy's course of analysis or Harary's graph theory, but also like the thousands of textbooks of all subjects that are continuously produced –, numerous mathematicians of all categories and levels of expertise have contributed, and are still contributing, to enlarging the background of resources made available for teachers to organize their instructional processes. And this background is mostly conceived in the paradigm of visiting works, where questions are made implicit in detriment to the answers elaborated.

Therefore, when teachers and students carry out a SRP, they do not always dispose of appropriate words, labels, discourses and narratives to talk about the inquiry process, to refer to its main elements and resources. This is especially the case when the inquiry process they live does not correspond to any of the work constructions $A^{\lozenge}$ elaborated by others. In these frequent cases, the elements of the Herbartian schema might appear as interesting tools, not only to design an analyze inquiry processes, but also to manage them in class with the students. The resources necessary to lead inquiries about questions that have not been studied before, that are not "important" or not considered at all in the mathematics domain, but that require the use of mathematical models and tools to be answered, these resources are not ready yet. Their elaboration needs the cooperation of scholars of all domains, involving in particular mathematicians working together with didacticians. I hope this International Workshop will be a productive encounter in this respect.

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# Struggle for the Next Stage of Higher Learning with Special Reference to Science and Technology

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**Abstract**: After the Japanese higher education system was rebuilt from the ruins of World War II, the graduates of science, technology, engineering, and mathematics (STEM) fields in the 1950s–1960s played important roles in Japanese industries, helping them to become world class. The academic performance of the graduates was also notable as indicated by the number of Nobel Prize winners. Recently, however, experts on higher learning are warning that the STEM fields in Japanese universities and colleges are in crisis. The decline is attributed to structural weakness of the Japanese universities and colleges caused by the delayed adaptation to the new stage of higher learning. This report focuses on undergraduate programs in the STEM fields and sheds light on the struggles for the next stage of higher learning to look for an appropriate way to reach a new goal. The historical perspective on the transition from elite to post-elite higher education and analyses of inconsistencies in undergraduate programs are followed by the proposition of the concept of "New STEM Courses."

**Keywords**: New STEM Courses, post-elite higher education, integration of sciences

The Harvard historian Professor Edwin O. Reischauer visited several universities to talk with students while he was the American ambassador to Japan in the 1960s. He had been shocked by the stormy student demonstrations peaking around 1960 protesting the revision of the Japan-U.S. Security Treaty. He soon realized that most of the students were making a bare living and thought that their dissatisfaction derived from their poverty. He was right in one sense, but he missed the sign of the explosive energy of students, and professors as well, after a prolonged period of isolation from the universities in the rest of the world. Information and immigration were strictly controlled throughout the war time as well as during the American occupation period. The students might have felt a great deal of dissatisfaction then but, on the other hand, they intuitively felt that a new era was close at hand. They spent a lot of time on political issues but, at the same time, aggressively engaged in academic activities as well. If the ambassador had been given a chance to visit one of the buildings of science or engineering, he would have found the vivid atmosphere of a learning community. Even undergraduate students were considered researchers in the laboratories and some of them actually contributed to solving the problems of their fields at that time. Among 23 Japanese and Japanese-born Nobel Prize winners in physics, chemistry, and biology, 13 were trained in the Japanese universities in this particular period, the 1950s–1960s. Needless to say, graduates of science, technology, engineering, and mathematics (STEM) fields played important roles over the next few decades in Japanese industries, helping them to become world class. The Japanese higher education system was thus rebuilt from the ruins of World War II.

The situation has since changed. In the period from1990–2000, the industry side began to question the qualifications of the graduates, especially those from the STEM fields. According to them, some of the recent graduates were not enthusiastic or assertive. At first, complaints carefully excluded the graduates from the traditional universities such as former imperial universities and prestigious private universities, but currently the distinction is being blurred. The university side also has recognized that a change took place in the students as early as the 1990s. Recently, Professor Yoshinori Osumi, a Nobel Prize winner in 2016, warned that science in Japanese higher learning was in crisis. Laboratories, especially those of the pure sciences, were suffering from the lack of research funds and, in addition, the young generation's interest was obviously leaving the STEM fields. In 2017, the British science journal *Nature* carried an editorial article warning that Japan was falling behind other leading countries, and the pace of research has slowed considerably over the past decade [1].

The author of the article in *Nature* as well as the Japanese media attributed the decline to the slowdown of the governmental budget increase for spending on science and technology in recent years. Meanwhile, other leading nations such as Germany, South Korea, and China have increased such spending significantly. However, if you compare the situation of the Japanese universities today to that of the 1960s described above, you will find that a simple increase in the governmental budget may not solve the problems. Rather, the decline should be attributed to structural weaknesses of the Japanese universities caused by the delayed adaptation to the new stage of higher learning. People in the STEM fields have clung to the customs and methods of traditional, elite-type universities and failed to cope with our knowledge-based, dynamically changing society. Although the problems are complicated and diverse, this report focuses on undergraduate programs in the STEM fields. By shedding light on the struggles for the next stage of higher learning, I wish to disclose some of the inconsistencies behind the formal pedagogical scheme and look for an appropriate way to reach a new goal.

## 1. Transition from elite to post-elite higher education

The undergraduate program of Japanese universities is conveniently divided into two categories: course work and laboratory work (lab work). Here, the lab work is defined as a burden assigned to each of the students to finish a graduation thesis, usually in the last year of the program. Traditionally, in Japanese universities, emphasis has been put on the lab work conducted in each laboratory, led by a professor. Credits for laboratory practice for each subject are counted as those of course work and not included in the lab work. Aside from the lab work, the course work has been criticized for a long time. The lectures are one-sided, the content is old and hardly revised, and examinations are too difficult or too easy depending on the lecturer, etc.

The contents of the examinations are not standardized, nor are they visible to those outside of the class because the professors conceive them to be an important part of "academic freedom." However, in society, doubt as to whether university students actually spend enough time studying to get the credits necessary for the bachelor's degree has grown to the point where it is not negligible. Lead by Professor Motohisa Kaneko of the University of Tokyo, a large-scale survey was conducted from 2005–2009, and it turned out that the speculation had certain grounds [2]. Outside of the classes, the average free study time of the students was substantially shorter than in other leading countries. "Free study time" refers to the time each student spends preparing for classes or finishing homework for the course. The term will be shortened here to "study time." Although there was controversy as to whether "study time" was an appropriate index, it has been accepted as one of the tools to determine the quality of a class. Based on the survey, the Central Council for Education suggested in 2012 that each organization should watch the students' study times. Back in the 1960s, Japanese university students were known as hard workers. What happened after that? Why did they stop studying seriously?

According to the statistics showing the changes of the study time by year, until the 1950s it was comparable to that in the pre-war era as indicated by Fig 1, despite big institutional changes. After the war, different kinds of higher education institutions, including liberal arts colleges, vocational schools, colleges of education and high schools were all reorganized into the four-year universities of "the new system.” As the students of vocational schools in the old system, for example, were known to work hard, the lack of a change in the study time in the new system was not surprising. In the late 1960s the study time started to decrease in correlation with the enrolment number, eventually coming down to the level of today, shorter than one hour per day. According to the model for higher learning proposed by Marin Trow, the enrollment number of 15 percent of the total number of a generation is a critical point in the development process. Below and above this point, “elite higher education” should be differentiated from “mass higher education.” The higher learning in a society where more than 50% of a generation go to colleges and universities is sometimes called “universal access higher education [3].” However, the features characteristic to this stage have not been recognized in the case of Japan, so I adopted the term “post-elite higher education” instead, which includes mass higher education as well as universal access.

Following Trows’ theory, it seems that the transition from elite to post-elite higher education started in the late 1960s. However, the change in the quality of learning was difficult to know then because of successive disorders caused by radical student movements in the 1970s. It was as late as the late 1990s when “the collapse” attracted attention from the public. A sudden decline in performance was observed in the 1980s by “fixed point observation” of the mathematical education in the Faculty of Engineering of the University of Tokyo, the top university in Japan. Students’ grades dropped by as much as 10 points out of a hundred in that decade [4]. Shocked by the decline, the Mathematical Society of Japan organized a working group in 1994 and conducted a survey. The results showed that the decline was occurring everywhere [5].

The universities could not take effective measures against the decline, partly because of their misunderstanding of the new stage of higher learning. The people of the universities knew that university education had become common, but considered that a kind of hierarchy still determined the quality of learning. In other words, they thought universities that were selective in terms enrollment

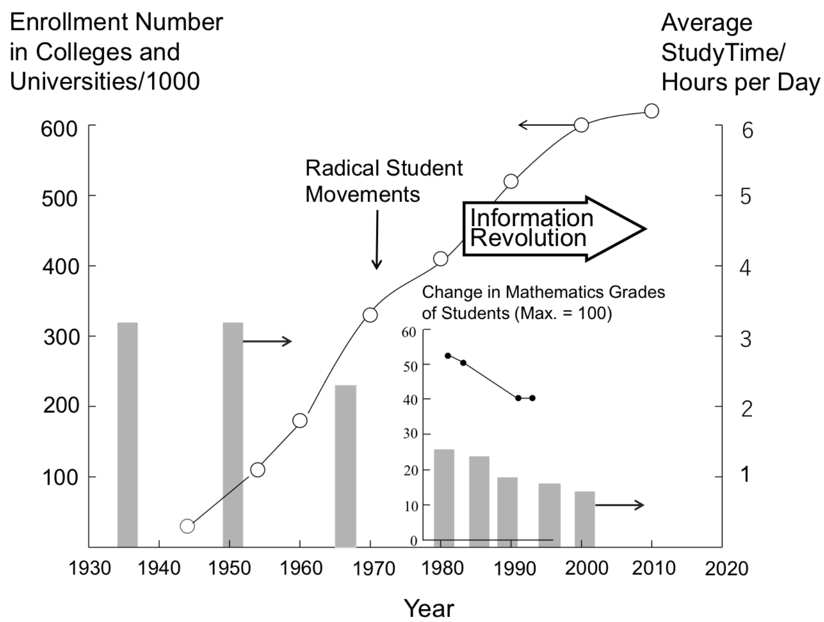


*Figure 1. Change in enrollment number in colleges and universities and average study time of students with year; the insert shows change in mathematics grades of students in the 1980s-1990s observed by “fixed point observation” of the mathematical education in the Faculty of Engineering, University of Tokyo.*

could sustain the traditional methods of teaching and learning, and did not need to change; only the low-ranked universities needed educational reform. Following this logic, most of the universities paid more attention to enrollment selectivity. In reality, the transition from elite higher education to the post-elite form occurred in a manner similar to a phase change in natural science. The quality of students changed rather suddenly in Japan from the top to the bottom as indicated by the insert in Fig. 1 and the surveys described above.

The change in students' attitudes was also caused by the rapid progress in information and communication technology in the 1980s, which led to the information revolution. Before the revolution, the universities had occupied a prestigious position by creating, monopolizing, and delivering academic information but, due to the revolution, at least the era of monopolization and the delivering of information was over. The paradigm shifted from knowledge-centered learning to knowledge-constructive learning, which induced a considerable change in the students' minds. It became clear that traditional lectures conveying knowledge to students were no longer effective and that a strategic reform should have been carried out in the 1990s at the latest. However, due to an accident in the history of the Japanese universities, the University Establishment Law was altered in 1991. As a result, the systems for liberal arts and science education in the former half of undergraduate programs had to be changed fundamentally. Every university was so busy coping with the new law that it had no time to deal with the new stage of higher learning. Thus, the important opportunity of the 1990s was lost.

## 2. Inconsistency in Undergraduate Programs

Most math teachers agree that the teaching and learning of mathematics in the colleges and universities is in crisis. This was not caused by internal or intrinsic problems of the field, but due to institutional changes made by the government. After the alteration of the University Establishment Law mentioned above, many of the programs for liberal arts and sciences for the first half of the undergraduate program were cancelled, resulting in the reduction of credits and the related professors' posts. The field of mathematics was not exceptional. The courses leading students to the mathematics majors barely survived, and others were considerably weakened. In many universities, there is no consensus about what the courses of mathematics for general purposes should be in the first half of the undergraduate program. The departments of engineering, for instance, put more emphasis on the informal and process approaches of mathematics, and pay less attention to the formal and structural approaches necessary for mathematics at the university level.

A growing defect in the undergraduate programs for social science majors is one of the serious problems. Because of the harsh competition in the entrance examination, the high school students choose either science orientation (*Rikei*) or humanities orientation (*Bunkei*) earlier in order to concentrate on the minimum number of subjects necessary for the entrance examinations. Historically, in high school, the socioeconomic orientations have been classified as *Bunkei.* The high school syllabus for mathematics in secondary education is strictly formulated by the government and, accordingly, *Bunkei* students can finish high school without selecting mathematics B and II: the two categories that teach exponential and logarithmic functions and vectors. Without mastering those basic tools of mathematics, it is difficult to study not only social sciences but also some of the humanities at the university level. Nevertheless, according to the recent survey by Professor Tetsuya Takahashi sponsored by the Japan Association for College and University Education, the universities, except for a few cases, are not dealing with the problems of those students [6].

The teaching and learning of natural science and technology in universities is also facing serious obstacles. Previously, *Rikei* students took it for granted that they should select physics in their high school days. They knew that without having a basic knowledge of physics the *Rikei* fields, including engineering, technology, medicine, pharmacy, etc., would be difficult to study at the university level. In the 1980s, more than 80% of the high school students selected physics, but the ratio declined to 34.4 % in 1991 and 33.7 % in 1993 [7]. Recently, the number is believed to be less than 30%. At the university level, even the courses of chemistry, biology, and earth sciences are designed with the presumption that the students understand the basic Newtonian concepts of physics such as acceleration, momentum, and energy, but this is no longer valid.

The discrepancy between the concept of "the university level" and the reality of students arises from two systematic contradictions. The first is the one involved in the entrance examinations. Different from the time of elite higher education, more than 700 universities and colleges are recruiting students today. The number of applicants changes in relation to the number of subjects set in the entrance examination, so the university side carefully narrows the subjects to the minimum number. This tendency is enhanced in the private sector because the number of applicants is one of the important factors in their financial management. The second contradiction is the big gap existing between secondary and university education. As far as the secondary education is concerned, everything is clearly formulated by the government as described above. However, the undergraduate programs of the universities are not standardized and depend on the classes. In this situation, high school teachers have nothing to do for the future of the students except for helping students prepare for the entrance examination, and the university teachers, on the other hand, have to treat students as if they all finished and mastered the secondary courses. These discrepancies should be eliminated in a systematic manner.

## 3. Concept of "the New STEM Courses"

The struggle for the next stage of higher learning is continuing. In response to the rapid decline of liberal arts and science education, the government created a new competitive fund to promote "good practices" in university education. The project was carried out by the Japan University Accreditation Association from 2003 through 2008 and the achievements were open to the public (8). In 2008, the Central Council for Education proposed a vision for the next stage of higher learning. Based on historical and sociological analyses, the report recommended rebuilding the undergraduate programs with special emphasis on new teaching and learning methods and the transition between secondary and university education. Along this line, excellent new ideas were put into practice and brought about the "educational reform movement" in colleges and universities in the last decade.

For example, a pioneering course on mathematics was started in 2007 in Ibaraki University, in which the newly entering students were classified into three grades, 0, I, and II, depending on the results of a test. Although the students classified into group 0 were judged not to have reached the university level in the beginning, remarkable progress was achieved after one year through a specially designed class bridging the gap between high school and university (8). The students acquired not only algorithmic skills but also learned to reason mathematically and use mathematical concepts and procedures at the university level. In Osaka Prefecture University, Professor Mitsuru Kawazoe and co-workers invented a new course in which, starting from questions arising from the world around us, students progressively gained the ability to deal with modeling processes and algorithmic skills [9].

Since 2008, I have engaged in creating "New STEM Courses" for the purpose of materializing new ideas, leading to a core concept in the first half of the undergraduate program [10]. The first goal of the courses is to bridge the gap between secondary and university education as exemplified by the two cases described above. The courses are helpful and useful particularly for first-year students.

The integration of science and technology, which are extremely specialized and fragmented at present, is the second feature of the courses. In an integrated science course invented in Tsukuba

University in 2009, discussion about the philosophical changes in modern science is followed by the great story of nature from the big bang to biological diversity. I joined this project as a chemist and delineated how elements and chemical species were produced in space and evolved into living things. The ingredients of chemistry such as the structure of atoms and the free energy changes due to reactions were built into the body of the course without any difficulty. The course's overarching essential concepts of modern science lead students to deep learning about the world around us. It is also helpful for the students who did not select any of the disciplines included in the course during high school, as it was designed so that the students gain a concept of modern science. This concept is also applicable to the field of technology. In Japan, technology and engineering are divided into many different departments but, at least in the first half of the undergraduate course, they should be integrated to help students understand the roles of state-of-the-art technology and engineering in the modern world.

The third feature of such courses is that they extend and join one discipline to another. For instance, in teaching and learning chemistry at the university level, the help of the mathematician is desperately needed. Without knowing quantum mechanics it is not possible to determine what atoms and compounds look like, but in the freshman courses, no mathematical base is currently provided. In dealing with the origin and evolution of life, cooperation between biology and geology is indispensable because the development itself has occurred via interactive effects of the two. Liberal arts and humanities could be combined with STEM so that the students can prepare for the rapidly changing society. It is time to eliminate the disciplinary barrier so as to meet the needs of the individual's current and future life as "a constructive, concerned and reflective citizen (OECD 1999)."

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# Mathematization:

# A Theory for Mathematics Curriculum Design

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**Abstract**: Mathematization, proposed by Freudenthal, H. (1973) for planning the curriculum under his re-invention principle, has been internationally known (Gravemeijer, K., Terwel, J., 2000). Before his proposal, Mathematization was already a principle of the Japanese secondary national textbooks in the middle of World War II and has been re-worded by several terminologies for the re-identifying mathematical activities on the aims of mathematics education. Under Isoda, M. (2012, 2015), this article clarifies the meaning of mathematization on the context of curriculum design, and also illustrates some significance of mathematization and the representation theory for designing mathematics curriculum from the school up to the university level with exemplars of the fundamental theorem of calculus.

**Keywords**: Mathematical Activity, Mathematical Thinking, Aims of Mathematics Education, Mathematics Curriculum

## 1. Why We Teach Mathematics

The question was proposed by Freudenthal, H. (1968) for explaining mathematization as an activity for teaching mathematics against the New Math movement. He defined mathematization by the re-organization of (mathematical) experiences (1973) by mathematical means. Instead of exploring why we teach mathematics, he objectified the activity of mathematization because he believed that the nature of mathematics should be taught through an activity.

The aims of mathematics education are clearly written in the curriculum standards of every country. *Figure 1* is the curriculum framework for ASEAN countries proposed from Southeast Asia Ministers of Education Organization RECSAM (Dom, Jahan, Isoda, 2017). In this context, mathematics education is a subject in school to cultivate human characters with the components of values, attitudes and habits of mind, ways of thinking and learning as well as the part of necessary knowledge in our life.

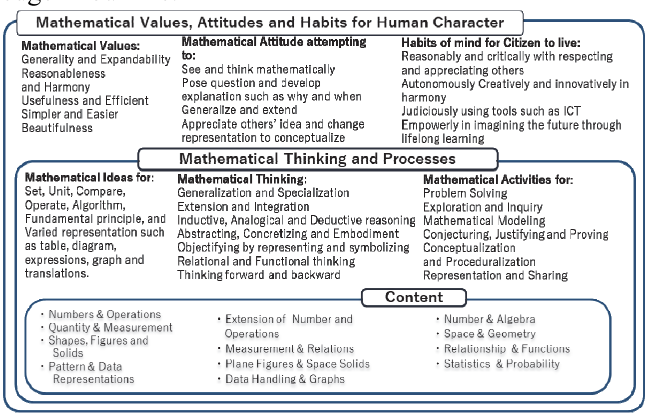



*Figure 1. SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) for Mathematics*

The ideas in *Figure 1* was also seen in the Japanese course of study (Ministry of Education, Japan, 1956. By Shimada, S.) in relation to developing mathematical thinking (see, Isoda & Katagiri, 2012).

For the achievement of the three aims of mathematics education as in *Figure 1*, mathematization is necessary because the activities can only be taught through activities. Thinking skills are usually learned through reflections of the learning processes and values are usually learned through the experiences of appreciation. Authentic activities provide the opportunities to learn those matter. For explaining the authentic activities, the terminology 'mathematization' is chosen to elaborate the principle for curriculum design.

## 2. Definition of Mathematization

Mathematization has been used as the theoretical basis for curriculum design. Freudenthal perspective of mathematics was used by Piaget, J. for his epistemology beyond contradiction and with logico-mathematical abstraction to establish new operations to operate existed operations. Later on, the Piagetian perspective of epistemology re-contextualized an exemplar of constructivism by Grasersfeld, E. v. and he defined the activity with viability. Viability in mathematics can be interpreted as developing the general and strong theories which are applicable. In Japan, the course of study in 1947, activities are explained by the re-organization of experiences on the context of Dewey, J. who was also referred as a constructivist. In 1968, the principle of mathematics curriculum was explained by the terminology of extension and integration. Even before Freudenthal, the process of mathematization was explained with the terminology of embodiment, abstraction of objects, and logical systematization in the Japanese teacher education textbook by Nabeshima and Tokita (1957) under the Philosophy of Mutai, R. (1947) (see, *Figure 2*).

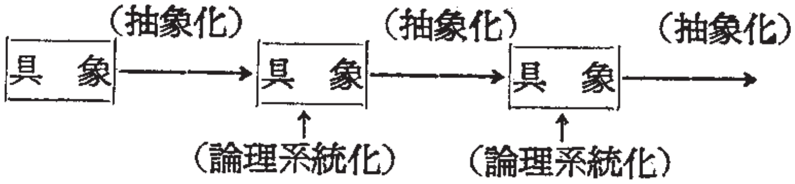



*Figure 2. Mathematization using the terminology of "[Embodiment]" and "→ (Abstraction)" with "(↑Logical Systematization)" by Nabeshima and Tokita (1957)*

Under these reviews of Freudenthal and a number of supportive articles including the articles of van Hiele, Isoda (1984, 2012, 2015) summarized Freudenthal's mathematization as follows:

1. Mathematization is the reorganization of experiences by using the mathematical methods.
2. The process of mathematization is described with levels:
   I. Object of Mathematization: Experiences are condensed through the activity of lower level mathematical methods.
   II. Mathematization: Methods of the lower level become the object of the upper level. Mathematical methods and experiences of the lower level are reorganized.
   III. Result of Mathematization: Experiences of the upper levels are condensed through the activities in that level.
3. Levels of Activity for living by Freudenthal and Levels of Thinking by van Hiele: Both of them referred levels to explain dis-continuity of learning process. Levels of Activities are described by the content of activity in relation to the organizing principle. Levels of Thinking are described as the difference of systems and languages with exemplar of van

Hiele Levels in Geometry. Both levels have the following features:

A) Every level has its own method in mathematics.
B) Levels of Activity describe the different mathematical intuitions and Levels of Thinking describe the different languages in mathematics.
C) Discontinuity: The difference of levels emerged as the contradiction or the difficulty of translations without appropriate terms for explanations.
D) Duality: The relationship between levels is the methods used for the lower level to become the object of the upper level.

At the Freudenthal Institute, his followers adapted his mathematization for designing curriculum in textbooks as a basic theory for designing curriculum. However they established their theory without levels (Gravemeijer, K. and Terwel, J., 2000). Freudenthal (1991) did not agreed to his followers' extended-terminologies. He re-enhanced and introduced his terminology of 'levels' for mathematization by re-paraphrasing levels as the world of living (see *Figure 3* by Isoda (1995)).

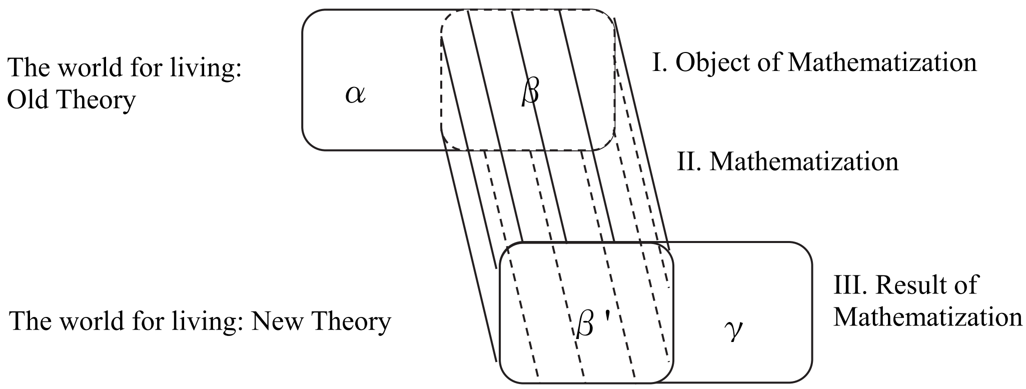



*Figure 3. Different World of Living and its Appreciation in Mathematization (Isoda, 1995)*

In *Figure 3*, the parts of α and β are the old theory, and through mathematization the part of β will be reorganized into β'. The correspondence appears translatable between β and β'. However there are inconsistency and need reorganization even though the same terms are used. The parts of β' and γ are the new theory. The part of γ illustrates the new theory to be appreciated which was never existed before. Even though the new theories are established, the old theory is still applicable depending on the context. Every level in the theory can be compartmentalized. The part of α has still meaningful after mathematization because such a part cannot be explained by the new theory.

The paradigm of the curriculum design given by Freudenthal is the discontinuity of learning process which explains the necessity of re-organization through mathematization. On this paradigm, school mathematics cannot be learned as in the New Math which try to construct school mathematics based on the sets and axioms. *Figure 3* illustrates the difference of levels as the difference of theories to show this discontinuity. However, it does not illustrate the process of mathematization itself. The terminology of mathematization by Freudenthal explains the sequence of teaching the content and its discontinuity but does not clarify the necessary activities in the process of mathematization. Due to this demands and limitations, Isoda revitalized Freudenthal's original framework for designing curriculum by establishment of levels for mathematization and explained the mathematization as the reorganizing process of the world of mathematical representations.

## 3. Representation Theory for Mathematization

Isoda established the representation theory for mathematization (Isoda, 1991). Based on his theory, every representation is defined by its symbols and operations. The changes of representation is referred as translation. If the using of symbols is consistent, it is called procedural translation. If the using of symbols changes, it is called conceptual translation. Even though every representation has meaning, the interpretable sequence of representations produce further context. Representation system is a set of such representations and their translations. The world of representations are established by a set of representation systems with some specific context. Isoda adopted those terminologies for analyzing mathematical activity for mathematization such as: analyzing the process of problem solving, the lesson studies on Number of Partitions (Isoda, 1987) the mathematical modeling of Crank Mechanism (Isoda & Matsuzaki, 2003), and the history of algebraic geometry including the Pascal's critique to Descartes based on Euclidian Elements. The reconstruction process of the World of Representations was deduced which describe the necessary activities for mathematization even though it does not describe the cognitive meaning of the real thinking process.

In the case of crank mechanism shown in *Figure 4*, the new symbol for representation is introduced on the existed representation world on clank mechanism [1] as a part of the existed world. At this moment, there is no operation for new symbol (a kind of mechanical structure) called "crank mechanism", but just a product of conceptual translation from manipulative mechanisms. It sets the rule for translation.

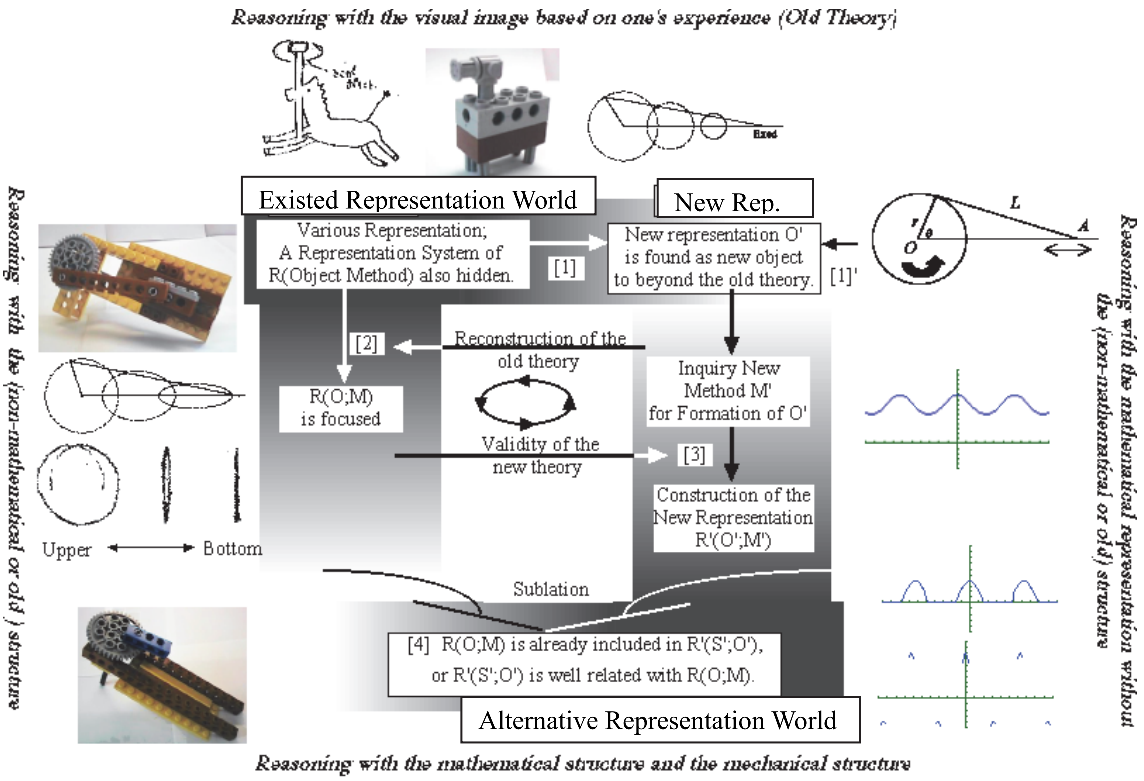
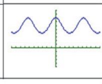
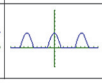
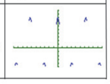



*Figure 4. Reconstruction process of the World of Representation through Mathematization.*

The crank mechanism [1] produced the equation of function, $f(\theta) = \mathrm{OA} = \mathrm{r}\cos\theta + \sqrt{\mathrm{L}^2 - 4\sin^2\theta}$, and a graph as unknown function as a representation with several variables. Those variables are the parameters to operate the crank mechanism. Thus, if the changing parameters are recognized as the operations of the crank, the graph of crank mechanism become the representation for the crank mechanism. For knowing the meaning of the parameters on the crank, the meaning of the discontinuous graphs are explored. The structure of the crank is explained in relation to the graph, then the operation of parameters becomes meaningful ([2] & [3]). Lastly, the odd or unusual (not continuous) graphs can be explained based on the original mechanism [4]. Then, we are able to explain the motion of the original crank mechanism from the graphs. The alterative representation world for the crank mechanism are subsequently established with the graph of function.

| Ratio | 1) $r : L = 1 : 3$ | 2) $r : L = 1 : 1$ | 3) $r : L = 3 : 1$ |
|---|---|---|---|
| Graphs |  |  |  |

In *Figure 4*, there are two gnomon (L-shape) which are separated parts of the whole square shape: The top gnomon corresponds to the activity of the lower level on the process of mathematization and the bottom gnomon corresponds to the activity of the next level. The motion of the Crank Mechanism was analyzed geometrically at the lower level. Through mathematization, the geometrical representation of crank mechanism which was a method of the lower level becomes the object for the graphs of trigonometric function.

The process of mathematization is generally explained by the theory of representations as follows: Firstly new symbol is introduced without its operation but with the translation rules in the existed representation world. At this moment, it is not considered as the special representation, but just a product of translation. Secondly, in the process of exploring the un-known operation of the new symbol (the right-bar part), related methods at the lower level (the left bar part) are focused. At this point, the method of the lower level becomes the object of exploration. Thirdly, once the operation of the new symbol (the method of next level) is established, new symbol can be operated without the old existed theory. Thus, the representation theory in *Figure 4* shows the process of how the methods is used for the lower level to become the object of the upper level. Firstly, the new symbol (β in *Figure 3*) is introduced with the lower level (α in *Figure 3*) and then, the unknown operations of symbol (β' in *Figure 3*) are introduced for the next level in relation to corresponding activity in the lower level. Finally, the focused for new operations of symbol (γ in *Figure 3*) produce the new levels.

For the curriculum design, the reconstruction process of the world of representation provides the task sequence which must be treated in the class: Firstly, how can we introduce new symbols under the existed representation world (the lower level)? Students should acquire the translation rule. Secondly, how can we explore the new operations to produce the symbols without translation? To do so, students need to translate for producing the symbols from the specific representation in the existed world. Thirdly, once the operations for the symbols are established, how can we shift to construction the alterative representation world without referring the existed representation world but involving the specific representations in the existed world as a part? Students should be able to reason with the new symbol and operations.

## 4. The Levels for Function up to Calculus (Isoda, M. 1984, 1995)

Until the New Math, there were various discussions on Geometry and Function (Calculus) to construct a better curriculum sequence likely from arithmetic to algebra. It was the reason why van

Hiele established thier geometry curriculum for his school by the Levels for Geometry until 1958[1]. Function up to calculus, under the Klein movement, the major issue was to integrate different subjects on mathematics into one integrated mathematics up to calculus. Hamley (1934) proposed the sequence with time and other quantity without connection to geometry such as the kinematics which enhanced by the Klein movement to integrate algebra and geometry as for the preparation of calculus.

Due to van Hiele already established van Hiele Levels for Geometry, Isoda established Levels of Function up to Calculus by using the general framework for the Levels with Hoffer-Isoda's work and justifying it by the evidences on history (epistemology) and students' development (genetic epistemology). The following is the comparative description of the van Hiele Levels for Geometry and the Levels of Function up to Calculus under the four conditions of levels: (A) to (D).

The Levels of Function up to Calculus is an extension of van Hiele Levels on Geometry to the other area which based on the basic difference of the structure of language which are used in every area of the school mathematics curriculum.

*Table 1. Discontinuity and Duality of Levels on Geometry and Functions (Isoda, 1996)*

| | The Levels of Geometry | The Levels of Function |
|---|---|---|
| Level 1 | Students explore matter (object) using *shapes (method)* | Students explore phenomena (object) using *unsophisticated relations or variation* (method) |
| Example of conflicts between levels | Because it has rounded corners, the road sign board 'YIELD' is not a triangle according to the meanings of Level 2, but we call the shape as a triangle in daily language. | In Japanese, we use "2 BAI, 3BAI" to mean "two times, three times" on level 2. But in everyday Japanese (Level 1), we can use "BA1" to mean either "double" or "plus". A child on level 1 says "BAI, BAI" ("plus plus") to mean three times the original amount. But "BAI, BAI" ("double double") usually means four times. On Level 2, students use "2 BAI, 3 BAI" to explain proportion as a covariance and they say three times as "3BAI" and do not say it "BAI, BAI". |
| Level 2 | Students explore the figures using the *properties*. The object on level 2 was the method on level 1. | Students explore the relations using *rules*. The object on level 2 was the method on Level 1. |
| Example of conflicts | A square is rectangular on Level 3, but not on Level 2. | The constant function is a function on Level 3 but 'constant' is not the relation which was discussed as covariation on Level 2. |
| Level 3 | Students explore the properties of figures using *implication*. | Students explore the rules using *notations of functions*. |
| Example of conflicts | The isosceles triangle has congruent angles. On Level 3, it is induced already and we do not have to explain more. On Level 4, we prove it. | On Level 3, a tangent line of quadrilateral function deduce using the property of only one common point/multiple root. On the Level 4, the tangent line does not always have this property. |
| Level 4 | Students explore the proposition, which is formed by implication, using *proof*. | Students explore functions using *derived or primitive function*. |

There are at least three significances to set the Levels for curriculum design. Firstly, it explains the appropriateness and inappropriateness of the curriculum sequence as for mathematization. Secondly, students' difficulty will be explained based on the difference of levels. For example, proportion in Japan has been treated in grade 5 and 4 as for the relationship of quantities on the Level 2 and

[1] Ministry of Education, Japan, also established the similar curriculum sequence for Geometry up to proof in 1958. Before van Hiele, Japanese such as Maeda also tried to establish similar sequence under the analysis of language in geometry.

redefined as the function with an equation at grade 7 as on the Level 3. On some tasks for propoetion for level 2, grade 7 and 8 students' achievements are lower than grade 6 students (Isoda, Shimizu, and Yamanaka, 1987). It is the evidence that students meet the problematic when they study upper Level content. They have to reorganize their knowledge under Level 2 to Level 3. The proportion on the Level 4 is the proportion for differential equations which is difficult to understand on the known proportion on the Level 2 and 3 because there are different usages of the constant in velocity and acceleration. Thirdly, if we already knew such difficulties, we can plan the curriculum for re-organizing and integrating the every related content of teaching beyond the difference of the Levels. For example, if we explain the Fundamental Theorem of Calculus as for organizing principle (Freudenthal, 1973) for every Level, sample activities for every Level can be resumed on Table 2.

*Table 1 Activities in Relation to Organizing Principle: the Case of Fund. Theo. of Calculus.*

| Level of Function | Explanation of Content with Activity for Fundamental Theorem for Calculus |
|---|---|
| Level 1 | Daily language: On the car, the acceleration on the speed meter is felt as the pressure to our back on the seat. Fill the water into the bottle. |
| Level 2 | Relations among quantities: Changes of the slopes on the line graph. Speed x times (Area on the graph) = distance |
| Level 3 | Algebra and Geometry: The rate of changes of various functions such as linear function, quadratic function and so on. |
| Level 4 | Calculus: Using the fundamental theorem of calculus. |

The activities on *Table 2* are known however it is not claimed as for teaching content of mathematics. Indeed, at the high school, the fundamental theorem is usually explained on the form of Level 4 and not clearly mentioned the relationship with other levels in the high school textbooks. Under the mathematization principle, those activities are necessary teaching content of mathematics.

## 5. Mathemataization for the Fundamental Theorem of Calculus

Based on the analysis of content on the Fundamental Theorem of Calculus by the Levels of Function and the Theory of Representations, Isoda and Seki (2008) developed the supplementary program for the low achievers of high school students. These students already learned the calculus on the polynomial function but only achieved algebraic operation of simple differentiation and integration without understanding it as an inverse operation. With this supplemental program (*Figure 5*), they were able to treat the inverse relationship between differentiation and integration through mathematization of their experiences designed from the program.

## 6. Further Discussion and Limitations

Matematization Principle for curriculum design supports the sequence of mathematization from the lower level to the upper level. However, at the upper levels, the order of levels can be set based on what the students had already learned. Thus, it does not always need to seek the sequence of learning from the concrete to the abstract. Curriculum sequence are usually explained by the strands or a net to explain it as the reticulate evolution with mutually related sequences. Mathematization shows a sequence belonging in a net as a necessary reorganizing process for discontinuity and construct mathematics beyond contradiction.

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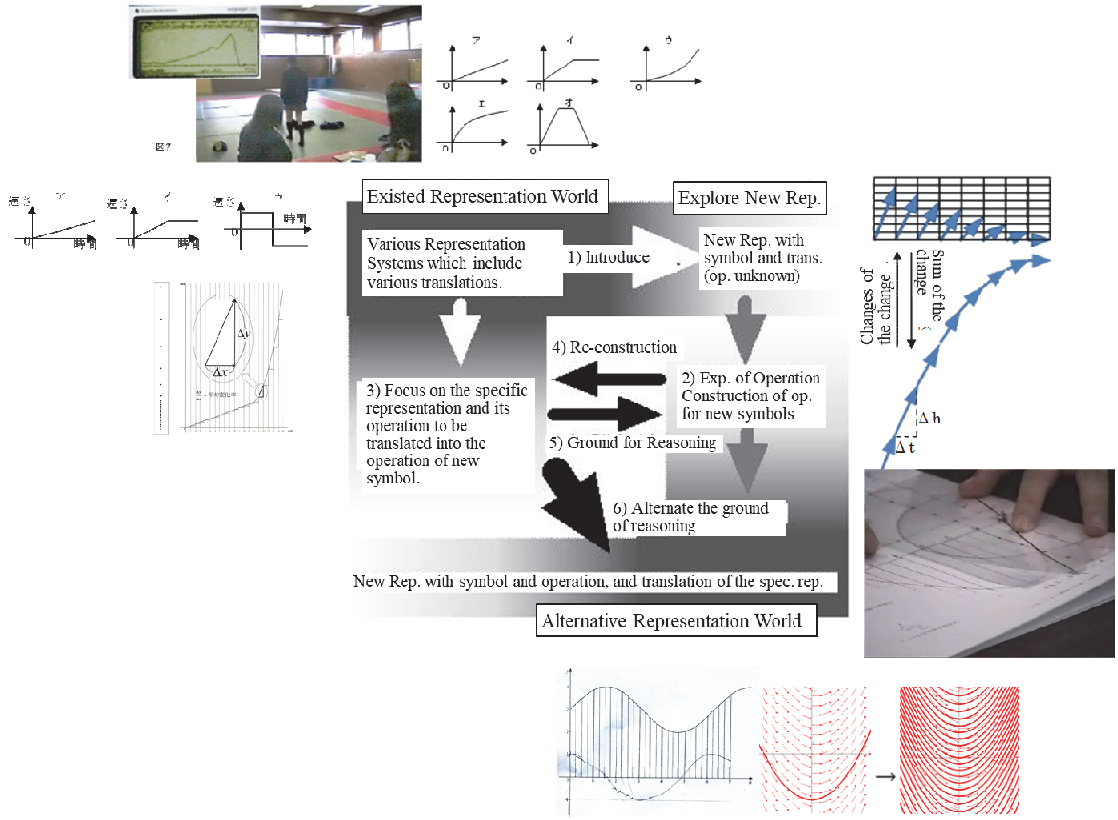


*Figure 5. Designed Curriculum for the Fundamental Theorem of Calculus under the Principle of Mathematization (Isoda & Seki, 2008; Isoda 2012, 2015)*

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# Statistical Thinking, Communication and Leadership

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**Abstract**: In knowledge-based society, it is shared understanding throughout the world that "Statistical Thinking" and "Competency of Statistical Analysis" is a substantial skill for detecting and solving new issues. "Japanese Inter-university Network for Statistical Education (JINSE)" has been newly organized in 2012. The first aim of JINSE is to develop Standard Curriculum and Teaching methodology for fostering human resources capable of coping with new issues, and eventually, to establish Quality Assurance system for statistical education by introducing Evaluation Committee consisting of members from academic statistical societies and other educational/economic organizations. In this paper, we introduce activities of this project and new trials for developing students' data analysis skills in Rikkyo University. One of our trials is to develop leadership and communication skills as well as statistical thinking.

**Keywords**: Data Science, Active learning

## 1. Introduction

Fostering people who can setup new challenging issues and solve them by applying data-oriented, quantitative skills have become essential to enhance industrial innovation in Japan in future. In knowledge-based society, it is shared understanding throughout the world that "Statistical Thinking" and "Competency of Statistical Analysis" is a substantial skill for detecting and solving new issues. Thus, building an educational system which aims to foster these abilities is internationally proceeding. Obviously, reinforcing statistical education is one of the most pressing issues for universities.

In Japan, Statistics is a hot topic in the media recently. Societies, industries, many places are interested in statistics and data sciences. However there are no departments of statistics in Japan now. Statistical educations are parts of each majors or general curriculum. Many of instructors may not be statisticians. Some of universities and Ministry of Education of Japan recognized the needs of revolutions of statistical education in Japan.

A new project of statistical education in Japan has been started in 2012. In this project, we have newly organized "Japanese Inter-university Network for Statistical Education (JINSE) ".The first aim of JINSE is to develop Standard Curriculum and Teaching methodology for fostering human resources capable of coping with new issues, and eventually, to establish Quality Assurance system for statistical education by introducing Evaluation Committee consisting of members from academic statistical societies and other educational/economic organizations

In this paper, details of this project and examples of new trials of Rikkyo University are shown and importance of developments of communication and leadership skills is also discussed. .

## 2. Japanese Inter-University Network for Statistical Education

Fostering people who can setup new challenging issues and solve them by applying data-oriented, quantitative skills have become essential to enhance industrial innovation in Japan in future. In knowledge-based society, it is shared understanding throughout the world that "Statistical Thinking" and "Competency of Statistical Analysis" is a substantial skill for detecting and solving new issues. Thus, building an educational system which aims to foster these abilities is internationally proceeding. Obviously, reinforcing statistical education is one of the most pressing issues for universities.

JINSE was originally organized as follows in 2012;

Eight Universities

The University of Tokyo, Osaka University, The Graduate University for Advanced Studies,
Aoyama Gakuin University (Head of the partnership) ,Tama University, Rikkyo University,
Waseda University, Doshisha University

Six Academic Societies

Japanese Society of Applied Statistics, Japanese Society of Computational Statistics,
The Biometric Society of Japan, The Behaviormetric Society of Japan,
The Japan Statistical Society, Japanese Classification Society

Eight Organizations

National Center for University Entrance Examinations, The Institute of Actuaries of Japan,
National Center for University Entrance Examinations, The Bank of Japan, Keidanren,
Japan Pharmaceutical Manufacturers Association, Japan Statistical Association,
Japan Marketing Research Association.

Shiga University joined JINSE in 2016 and started Faculty of Data Science in 2017. It is the first undergraduate data science course in Japan.

Goals and objectives of JINSE were as follows. We would foster college graduates with problem solving capability needed by the society. For this purpose, we established standard curriculum system for statistical education at the higher level, and implement standard performance measurement for assurance of statistical education.

Some of the participating universities started "Sub program" or "Minor program" for Statistics both at undergraduate and graduate level, by utilizing teaching materials provided by JINSE.

JINSE was a five year program for Promoting Inter-University Collaborative Education supported by Ministry of Education, Culture, Sports, Science and Technology (MEXT). When the original JINSE project was over, we have accumulated resources of teaching material and assessment method in JINSE. JINSE can offer them for all universities in Japan, which could enable us to perform statistical education to fill the needs from our society. In 2017, an extended JINSE started as a renewal of the original JINSE. Any university and institutions can join the network.

## 3. Examples of New Trials in Rikkyo University

Rikyo University is one of eight universities in JINSE. Rikkyo University launched a new center for statistics education, survey research and data archives, named the Center for Statistics and Information (CSI), in March 2010. In Japan, there are no departments and faculties of Statistics. The demands for statistics education and consultations for data analysis, however, are very strong as like other countries. A survey was conducted by Senuma [1] to determine what students were expected to study through mathematical studies at universities. This survey was conducted on all the companies listed in the Tokyo Stock Exchange. The results revealed that statistics education, which enables students to use data substantially, is regarded as highly desirable.

Watanabe and Yamaguchi [2] reported the process of developing the e-learning contents and educational materials for statistics education. Watanabe and Yamaguchi [2] also pointed out the needs for changing the classical styles for statistics education as follows; numerous statistics teachers in arts departments are of the opinion that students, in general, are hesitant to study the type of statistics that emphasizes mathematical aspects. Course materials utilizing the Internet and other multimedia resources have recently been developed and put to practical use in university education. Multimedia materials emphasize audio and visual components that can be interactively operated and verified. It is hoped that the use of multimedia will positively affect university education; however, no concrete lecture form that will create that positive effect has been standardized in the field of statistics. One of the possible reasons for this failure is that most of the syllabuses that are publicly available are developed in text form and are not based on Internet awareness or the course materials being converted into multimedia formats.

On the other hand, Utts [3] and the GAISE report of American Statistical Association suggest a new style for statistics education and contents students should learn in higher education. CSI in Rikkyo University provides e-learning courses for social survey and basic statistics, which are developed on the according their suggestions.

CSI started to provide four subjects in 2010; “Introduction to the Social Survey”, “Social Survey Methodology”, “Introduction to the Statistics: Descriptive Statistics” and “Introduction to the Statistics: Statistical Inferences”. Hirose et al. [4] introduced details of these courses. “Introduction to the multivariate analysis” started in 2011. Students can learn about survey methods, for example, designs of samples, how to make questionnaires and so on, as well as basic statistics in this course. The maximum class size of each subject is two hundreds and expected size of each class may be 150. All students in Rikkyo University can take these subjects.

These five subjects have been accredited by Japanese Association for Social Researchers as the course for social researchers. The association has been established by the following three academic societies, the Japan Society of Educational Sociology, The Japan Sociological Society and the Behaviormetric Society of Japan. The association has a leadership for the social survey education in Japan. On the other hand, the Japan Statistics Society has special committee of statistics education. We can get many information and ideas on education on statistics from the committee. In a sense, our course is supported by them. In 2018, Rikkyo University will start Data Science Minor course, which can be taken by all students in undergraduate course in Rikkyo University.

## 4. Statistical Leadership And Communication skills

The Business Leadership Program (BLP) is the core curriculum of the Department of Business, Rikkyo University and encourages students to take an active role in the global community. Through team-based projects and skill-enhancing exercises, BLP nurtures business leadership capabilities in an experience-based learning environment.

The Business Leadership Program (BLP) begins with an "Introduction to Leadership" course in the Spring semester of the first year, and concludes with BL4 in the Spring of the third year. This five-semester course of study has a dual approach, using project implementation and skill enhancement to develop leadership. In the semester dedicated to project implementation, students learn to recognize their strengths, and in the succeeding semester dedicated to skill enhancement they work to develop these good points intensively. The cycle continues in the next project implementation semester, where students can check their own progress. Rikkyo University started the Global Leadership Program (GLP) for all students, which was extended program from BLP in 2013. Statistical skills are very important for working on real problems. So students in BLP can recognize the importance of statistical knowledge and thinking.

Importance of Statistical leadership is pointed out by Snee and Hoerl [5]. Rodriguez [6] gave three comments on the statistical leadership as follows. "First, the road to statistical leadership begins with volunteering. Second, successful leaders work on their communication skills and apply them as champions for our field. Third, great leaders encourage and develop younger leaders".

In BLP and GLP, an action learning method is used for developing leadership skills. Action learning is a process which involves working on real challenges, using the knowledge and skills of a small group of people combined with skilled questioning, to re-interpret old and familiar concepts and produce fresh ideas (see Revans, [7], [8]). This method can be combined into group works on statistics classes. A combination program of statistics course and leadership program was started in 2016. This program was planned for developing students' problem solving skills using statistical skills as well as leadership skills. Details will be shown in the presentation.

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# Research Reports

## 研究実践報告

# Advanced Mathematical Literacy and Designs of 1st Year Mathematical Courses for STEM Students

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**Abstract:** Defining AML as a competence, we first focus on the values of mathematics and mathematics education. Proposing some answers to the question of the values, arguments proceed to the way to encourage students' autonomy. Finally, examples of epistemological designs of linear algebra are shown. We conclude what is important is a construction of epistemological "world" being aware of values of mathematical education and students' autonomy.

**Keyword:** mathematical literacy, value, epistemology, autonomy

## 1. AML

Let us begin with explaining "advanced mathematical literacy" (AML). We here restrict ourselves mathematical education in non-math major, at least temporarily.

The term AML comes, of course, from mathematical literacy defined by OECD - PISA. In its definition of mathematical literacy, the level of the mathematical knowledge is formally not limited to that of 9th or 10th grade. On the other hand, M. Artigue pointed out in her concluding paper to IWME-2014[1], that "the idea of mathematical literacy is understood and used in the educational world quite differently" and if one use it in the secondary-tertiary transition, it should be "an advanced vision of mathematical literacy", which is considered at least as demanding as regular mathematics courses. Accordingly, we added "advanced" to distinguish our concept from that of PISA's.

Artigue[1] proceeded to state that the idea of mathematical literacy is used for questioning teaching practices that do not allow students to access the *raison d'être*. Further she noted the possibility such use of "mathematical literacy" is related to modelling and applications or inquiry-based practices in mathematical teaching. We have just intended to develop such practices and moreover wished to use this concept (AML) to express the goal of mathematics education at lower years in non-math majors. Henceforth we have intended to interpret AML as a competence adopting the definition of EU [2]:

> *a competence refers to a complex combination of knowledge, skills, understanding, values, attitudes and desire which lead to effective, embodied human action in the world in a particular domain,*

which was originally given by Hoskin & Crick[3]. We note that the following explanation [2]:

> *Importantly, competences are expressed in action and by definition are embedded in narratives and shaped by value・・・・.*

So we expect students to acquire AML as a competence so that they use mathematical concepts and skills fluently in their future business and also take actions in the shapes of reason and truth, and behave with constructive, engaged and reflective attitude as citizens, as is described in the definition of mathematical literacy by PISA.

We note that the term competence is also used in the context of higher education in Tuning Project in EU or AHELO in OECD. In such cases, the goal is to develop qualification of higher education, compatibility of units, and to measure the levels of abilities of students. So, non-cognitive components such as values and attitudes are supposed to be eliminated in the definition [4].

*Competences represent a dynamic combination of knowledge, understanding, skills and abilities.*

Since our aim is to reform and to develop mathematical education, it seems to bring fruitful results to reconsider values of mathematics and mathematical education. Thus we define AML as a competence defined in the above manner and are going to argue the values.

## 2. Values of mathematics and mathematics education

To our aim, it may be highly necessary to specify the value of mathematics and mathematics education to develop students' AML. As to value of mathematics we can quote a report by a committee concerning mathematical education of Science Council of Japan [5], which shows three categories values:

V1: having practical uses: in daily life, in natural sciences, in engineering, in social science and also humanities, in digital technology,

V2: Nature of mathematics as culture : logical rigidity, integrity, beauty, history of mathematics has been made as results of human efforts and wisdom,

V3: Cultivating characters - mental characteristics gained by studying mathematics: logical reasoning, concise expression, how to see comprehensively

As to value of mathematics, the celebrated work of A. Bishop [5] ought to be mentioned. In this book Bishop listed up 6 values of mathematics that can be judged belonging to V2 and V3. He also emphasized the practical use of mathematical knowledge and put it at the top of three components of his culturalization curriculum. Nevertheless, he didn't count practical use as value. In his terminology, value means spiritual tendency and does not mean usefulness for technology. By value we mean every important and precious features and functions. The difference between Bishop and us lies not in evaluation of mathematics but in the terminology of "value." There might be no doubt that the value of mathematics falls suitably into those three categories V1 to V3.

To argue mathematics education in non-math majors to develop AML, I suppose it is necessary to clarify values of mathematical education in non-math majors by some descriptions. I suppose, the value of mathematical education is to make student aware of specified value of mathematics knowledge and to induce them to accept and internalize it. I have not enough conviction indeed, but I propose the followings as values of mathematical education in non-math majors: it is to make a starting point of discussion:

V1': to let students understand mathematical knowledge useful for practical use, to let them how to use this and also let them acquire the general usage of this,

V2': to let students aware of values of the nature of mathematics: abstract, logical, structural, general, real, explicit, exact, precise, truth, intuitive, imaginable, etc.,

V3': to contribute to cultivate students' characters expected as a citizen:

These values are listed to let student willing to study mathematics, to use mathematics in their major, to confirm the grounds of mathematical knowledge they use, and for them to acquire values and attitudes evaluated wise and intelligent in the society. We note that values V1' and V2' can be meaningfully introduced in the learning of mathematics, but V3' will be meaningful only together with the similar values of other subjects including social sciences and humanities.

One of the reasons to insist the importance of value of mathematics education is to overcome "applicationism". Here, applicationism was introduced by Barquero, Bosch and Gascón [6], [7] as having a high degree of agreement with the five indicators including

$I_1$: *Mathematics is independent of other disciplines.*

$I_4$: *Applications always come after the basic mathematical training.*

$I_5$: *Extra-mathematical system could be taught without any reference to mathematical models.*

They showed in these papers that applicationism is the dominant epistemology in natural sciences. They concluded that applicationism establishes the separation of mathematics and natural sciences: but situation is the same in majors of technology and engineering.

Applicationism could be interpreted as a conflict in values, I suppose. Mathematics teachers may not really deny the value V1 but they don't want to spend his precious class time to exhibit values of V1 at the cost of exhibiting values of V2, and they never doubt the validity of $I_4$. Staffs of other majors observe this and also observe the difficulty their students suffer in such mathematics courses, tend to say "Mathematics is of no use. It suffices to teach mathematical skills useful for our major." at least in Japan. So we should stand at the position of the third party, of the society or the public, and specify the values of mathematics education that could really develop students' competences including what is expected as the results of their major.

I hope that being aware of the values of mathematics educational may be a possible way to overcome the applicationism. But how we can really do it?

## 3. Designs for mathematical education

I have explained that mathematics teachers should clearly aware of values of mathematics corresponding to the knowledge they treat in their course. At the same time, it ought to be stressed that teachers should also be aware of students' autonomy. Referring the theory of constructivism, knowledge must be constructed by the learner oneself. If one is not willing to consider positively in his learning, one never constructs knowledge so that one cannot understand or meaningfully solve something. Understanding of mathematical knowledge cannot be achieved by simply hearing explanations even if the teacher's skill of explanation is very well. Students have to think autonomously by themselves to understand or solve something.

To encourage students' autonomy, it is recommended to follow Chevallard's "questioning the world" paradigm [8] as was introduced by Artigue [1]. Several designs has been developed to follow that paradigm: ASR, PSR, and also SRC, RSC, which was proposed by Chevallard and C. Winslow [9][10]. One can add "guided reinvention of mathematics" design in RME that came from the idea of H. Freudentahl. Those designs are all consider field of mathematical knowledge not as fixed and completed world, but as an open, living and developing world full of interesting problems waiting someone to solve. Here, designing a course of mathematics is to prepare this world of knowledge and problems students can walk about using teacher's advices and supports to reach the destination at last. This means an epistemological reform no matter how small it is. In most cases, the path students follow may be, regretfully, a fixed one. At some points of the path, teacher prepare exercises to attain deep understanding of a concept or a proposition or to master a procedure. So called "praxeologies" are often used – a sequence of exercises to get intuition of a structure of a theorem or of other knowledge, divided into praxis part and theoretical part [10]. A teacher, after preparing this world, explain outline of the world and a local geography they stay now, give advices, then encourage them to advance to today's destination. In a word, a teacher is expected to play a facilitator and a tutor providing learning supports if necessary. At an appropriate time, a teacher gives additional explanation, concludes a step and proceeds to the following one.

Among the all tasks of a teacher, the most important one is to construct the world with no doubt.

After maintaining a lesson, one should reflect today's affairs, evaluating students today's work and modify the world if necessary. In this manner, the world is improved reflecting students' real ability, attitude and desire. It may need several year to improve the world just as one agree with oneself. However, one can feel free to start this process with some idea of the world come put by oneself or by someone else. This is the way I make up a small world in linear algebra in 2014-2017, examples of which are introduced in the following section.

Another attention is appropriate. A design of course, lesson and self-learning referring theories of instructional design may considerably improve the course, which will be explained the following speaker, H. Komatsugawa.

## 4 Examples of the world in linear algebra

Let us introduce the "world" developed in the course of linear algebra of my belonging institute. The course consists of 15 lessons of 90 minutes. The number of the students was 12~15 varying by year. They were not so good at but like mathematics. Nearly half students were aspiring teacher of secondary school. Through the course, a question "what happens to solutions of systems of linear equation" is consistent. Here I introduce a bit of two topics "linear independence/dependence" and "determinant". After letting students understand the rank, the lessons enter the former topics.

### 4.1 Letting students to understand the concept of linear independent/dependent

We begin this topic with the following question: "Why some matrices like $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & -3 \end{pmatrix}$ has a rank smaller than numbers of rows? In other word, why and how the 0–row appears through row elimination in a matrix originally having no 0–row?" After giving some minutes to consider this problem, an advice is given if necessary: let the matrix $(A|A) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & \boldsymbol{a}_1 \\ 1 & 3 & 1 & \boldsymbol{a}_2 \\ 1 & 1 & -3 & \boldsymbol{a}_3 \end{array}\right)$ be transformed by elementary row operations into a matrix with E as the left half. Students can obtain $(A|A) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \boldsymbol{a}_1 \\ 0 & 1 & 2 & \boldsymbol{a}_2 \\ 0 & 0 & 0 & \boldsymbol{a}_3 - 2\boldsymbol{a}_1 + \boldsymbol{a}_2 \end{array}\right)$ concluding $\boldsymbol{0} = \boldsymbol{a}_3 - 2\boldsymbol{a}_1 + \boldsymbol{a}_2$. By this students have insight for linear dependence. Importantly, this concept appears in a meaningful context with sure reality. From here a teacher can define correctly and intuitively linear independence and dependence. After then, other examples should be given: examples of linear equations, geometric vectors, matrices with and without inverse. Several exercises can be properly questioned if there is enough time: the followings are to prepare proving multi-linearity of the determinant from certain properties.

Q1: Prove 3-tuple of 2 component vectors is linear dependent.

Q2: Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are 2 component row vectors. Show a matrix $\begin{pmatrix} \boldsymbol{x}+\boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$ can be transformed, if $\boldsymbol{z} \neq \boldsymbol{0}$, into $\begin{pmatrix} \alpha\boldsymbol{x} \\ \boldsymbol{z} \end{pmatrix}$ or $\begin{pmatrix} \beta\boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$ where $\alpha, \beta$ are some constants.

### 4.2 To give intuitive definition of Determinant and to give several ways of correct definition

The lesson begins with a question "You know a square matrix $A$ has an inverse iff the set of all row vectors is linear independent. Can you express this condition by an expression of components of $A$?" To answer this question, three or more lessons will be necessary. Lessons are constituted with several small questions $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3, \cdots$.

Q1: What happens when the number of components is two?

Answer: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $d(A) = ad - bc$. Then $A$ has an inverse iff 2-tuple of $(a\ b)$ and $(c\ d)$ is linear independent, which implies $d(A) \coloneqq ad - bc \neq 0$.

Q2: What happens when the number of components is three? (There will be no answer.)

Advice to show a strategy: There are significant properties of $d(A)$. Can you find them? If you can find them, you might calculate a similar function $d(A)$ for a 3-order matrix.

Further advice: Let $T$ be an elementary matrix. Compare $d(TA)$ with $d(A)$.

Q3: Can you calculate $d(A)$ for a 2-order square matrix $A$? If not, find another property so that you can calculate $d(A)$ using properties you have found.

Q4: Can you calculate $d(A)$ for 3-order square matrix $A$?

Exercises for the calculations:

Q5: Compare the value of $d(A)$ the cases $A$ has and not has an inverse.

(Note that till now students have mastered calculation of $d(A)$ for 3-order matrices. They will be easily able to extend the result for 4- or any order matrices. )

Q6: Give an expression of $d(A)$ for 3-order square matrix $A$ with its components $a_{ij}$?

Advice: The following properties are caller multi-linearity. One can prove these from the properties of $d(A)$ you have found. Use multi-linearity to find the expression.

Further advices for expanding $d(A)$ and $\cdots$

Further questions to proceed to show convenient properties of the determinant, and to introduce permutations and the sign of permutations

## 5. Conclusion

For the development of mathematics education in non-math major, we propose to focus on values of mathematics and mathematics education. Though theoretical progress is not yet enough, it could be said that practical research has gotten a result. Designing such education, what is the most important is to construct epistemological “world”, even if that is so small, under awareness of values of mathematical education and students’ autonomy. With such education, development of students’ competence AML is hopeful not to yet say reliable. Teachers are expected to act as a facilitator and to improve the “world” taking advantage of the experience. The course design should be assisted by one more design referring ID theory. Along with accumulation of development of the epistemological “world” and experience in the classroom, theoretical progress especially on the clarification of value is strongly expected.

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## 高水準の数学的リテラシーと理工系初年次数学科目のデザイン

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著者たちはコンピテンスとしての高水準の数学的リテラシーの育成を教育目的として研究と実践を行っている。これは特に価値を重視するということであるが，まず数学及び数学教育の価値を明確にすることが大切である。著者たちは，学術会議の数学・数学教育グループが出した提言から数学の価値を参照し，更に数学教育の価値を期待する学生像と重ね合わせて考察している。価値を重視する一つの理由は，バルケッロらの「アプリケーショニズム」を克服することで，この傾向の発生はそもそも数学教育の価値に関する対立が根底にあると考えるからである。価値をまず客観的立場から考え，その価値を意識した教育によってこの傾向が克服されることを期待している。教育をデザインする際には，この価値と学生の自律性を意識して，数学的知識を組み立て直すことが最も大切であろう。自律性を養うには，問題・助言・解答を組み合わせて最後の到達点にまで達することができるような，一つの知識「世界」を作ることが望ましい。これは，シュバラールらの提起等から始まった方法で，幾つかのデザインが提案・実践されている。その「世界」の中で学生と教員が知的な交流を行なうことを望んでいるのであり，授業では教員はファシリテータとして振舞うことが望ましい。その「世界」は年々の教育実践によって改良していく。なお授業と自己学習を併せた，ID 理論を参照した教育デザインによる補強が強く望まれる。

著者が作った「世界」の例として，線形代数の授業から 1 次独立／従属概念と行列式の世界の一部を挙げておく。

実践経験の豊富化，構成された知的「世界」の蓄積と並んで，理論の進展，特に価値の明確化が強く期待される。

# A Model of Flipped Classroom Using an Adaptive Learning System

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**Abstract** : In this paper, we propose a model of flipped classroom using an adaptive learning system that provides a function of computer-based testing and Training measuring the degree of understanding through the item response theory. We have evaluated the learning effectiveness of this system through a case study of a C programming class.
**Keywords**: flipped classroom, computer-based testing, computer-based training

## 1. Introduction

It is important for learners to acquire knowledge in academic disciplines requiring advanced expertise, and it is necessary to confirm the degree of their knowledge in various situations of lectures. In a previous study, some of us developed a learning support system with a knowledge map in which domain knowledge was categorized and structured (Takano et al., 2014). Using this system, learners can view knowledge items on the map and solve quizzes related to the considered item. We evaluated the system effectiveness for the acquisition of basic knowledge through a case study. In the present study, we extend the system to an adaptive one that can provide learning quizzes corresponding to learners' understanding degree of knowledge determined on the basis of the item response theory (IRT). The system mainly provides two functions, namely, (i) an adaptive test function (ATE) and (ii) an adaptive training function (ATR). We propose a learning model of flipped classroom using the system, which has been evaluated through a case study in a C programming class.

## 2. Proposed Model

### 2.1 System

In our constructed system, knowledge items on the knowledge map are linked to learning quizzes corresponding to that knowledge[1]. Note that the quiz format comprises a problem, an answer, and an explanation. Each learning quiz is classified into seven levels of categories determined by the IRT. In the first learning step, both ATR and ATE functions start to provide learning quizzes at level three and change quizzes adaptively, corresponding to the value of the learner's learning ability based on the IRT. The ATE function provides tests in which problems are automatically selected from quizzes. In the ATR function, the quizzes are adaptively provided on the basis of the learner's history of correct or incorrect answers through the knowledge map.

### 2.2 Learning Design

We assume that several learning objectives are applied to a class and several lessons in the class are needed to master one objective. For instance, we have fifteen lessons in the C programming class, and three lessons are needed to master the ability for using functions in the C programming language. We define the period of the lessons for mastering the given learning objective as a "learning unit". In the present study, the learning objective is related to a knowledge item in the map and learning quizzes included in the item are structured in accordance with the understanding degree of the learning objective. As mentioned above, the learning quizzes are classified into seven levels depending on the IRT, and according to the levels, the objective of understanding knowledge concept is set to level 1~2, that of knowledge utilization is set to level 3~5, and that of application of knowledge is set to level 6~7.

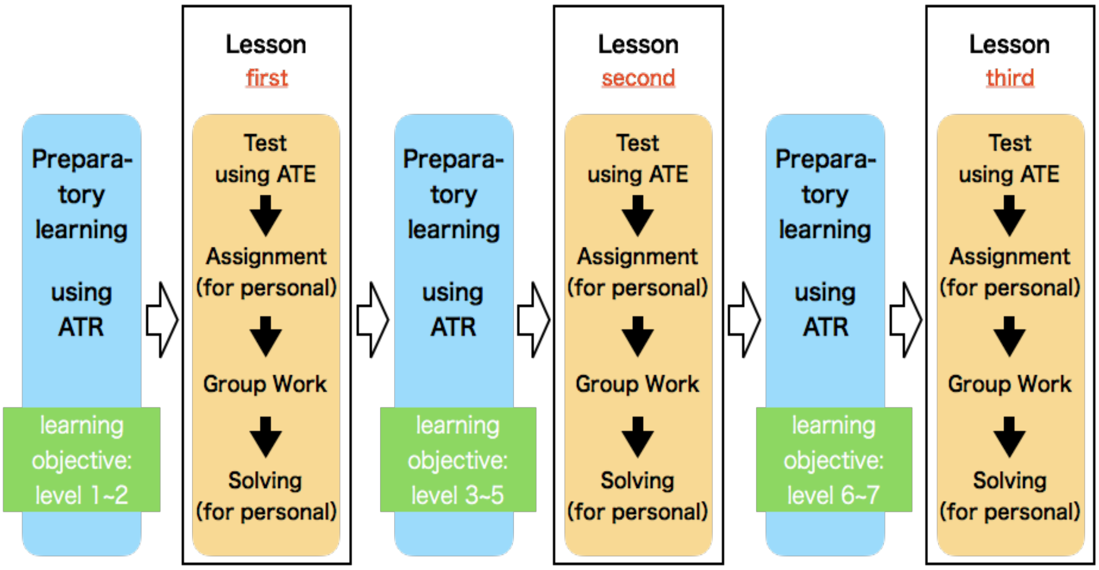


*Figure 1. A Model for a Flipped Classroom in C Programming Class.*

## 2.3 Case Study

We introduced this model to a class on C programming. The scheme of its "learning units" is shown in Figure 1, and we suppose that each one of them consists of three lessons. The learning objectives of each lesson correspond to those of level 1~2, 3~5, 6~7, successively. Before each lesson, learners are recommended to do preparatory learning and for this step they can use the function of ATR. At the beginning of each lesson, learners are assigned to take tests for checking the degree of understanding for the preparatory learning. The first characteristic of our model lies in the system utilization. The tests are performed using the function of ATE and the data of learning degrees are automatically stored in the system and the teacher can gauge all learners' progress and control or manage the classroom easily. For instance, when the teacher plans to coordinate group work in the classroom, he or she can consider all members' learning ability in each group. Then it is expected that the learner with the highest score in the group encourage all other members in his or her group and advise them to actively participate in the group work. The second characteristic of our model lies in the capability of managing various learning situations through the iterated learning process by using the system. All learners start preparatory learning in the first step of the "learning unit" shown in Figure 1, using ATR out of classrooms. In the middle step, some learners also do the preparatory learning for the second class, but others may review the first class because of their lack of knowledge. In our learning model, this learning phase is allowed by the use of ATR. Learners are adaptively recommended to do their exercises and gain total knowledge in the "learning unit" through our implemented system.

## 3 Evaluation

### 3.1 Evaluation of ATE Results and Midterm Examination

We performed a case study of a C programming class in which 7 lessons were held and classified into three "learning units". The objectives of each "learning unit" were "Variables, If and Loops" in the first 2 lessons, "Variables, If, Loops and Arrays" in the third and fourth lesson, and "Functions" in the last 3 lessons. Results of ATE in each lesson are shown in Figure 2, which shows that the understanding degree of each learning objective is improved at the end of each "learning unit". This result indicates the effectiveness of our learning model using this system.

We performed the same midterm examination as last year, which was not adapted to our model; the number of students was 75 in 2017 and 74 in 2016. We found that the number of students with a score lower than 70 decreased from 15 to 4 and the number of students with a score higher than 80 increased from 45 to 60. Our result indicates that the learning model using CBT contributes to the improvement of the learners' degrees of knowledge.

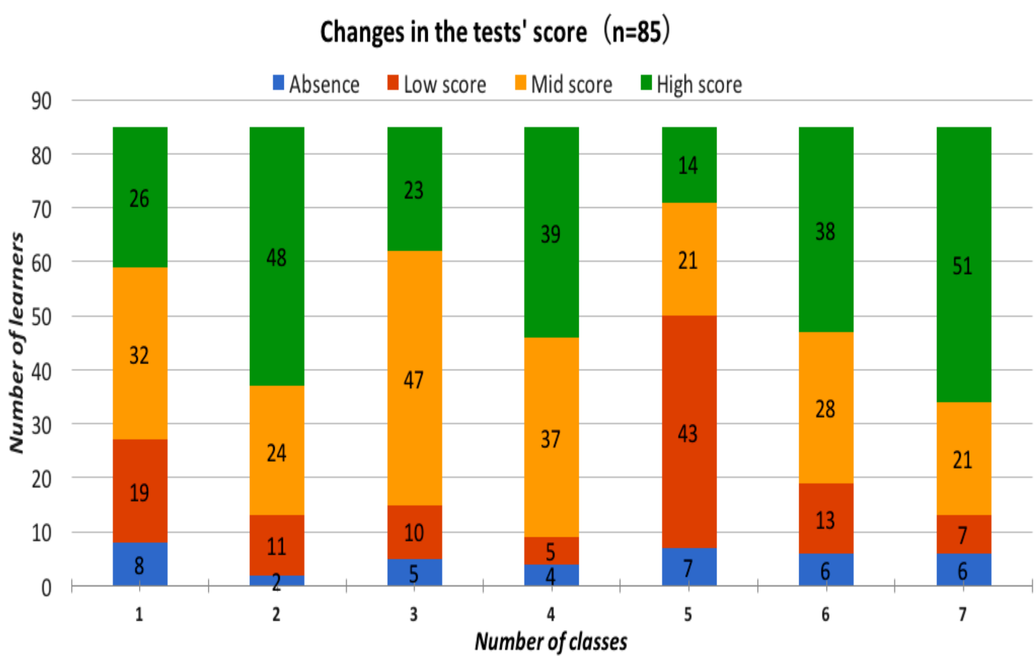


*Figure 2. Changes in the tests' score.*

### 3.2 Evaluation through Questionnaires

Three questionnaire surveys were conducted with the students. In the first questionnaire with 62 students, they were asked to answer the question: "Do you think you have trained your own ability (think for yourself and try to solve the problem) in the class?" Approximately 84% of the students answered "Yes". In the second questionnaire with 62 students, they were asked to answer the question: "Do you think it is good to be able to confirm your level through the test in each lesson?" Approximately 84% of the students answered "Yes". In the third questionnaire with 72 students, they were asked to answer the question: "Do you think it is useful for you to confirm your own level through the tests and learn using the knowledge map on improving your programming skill?"

Approximately 88% of the students gave a positive response. The result also supports the notion that our model gives a positive contribution to the improvement of the degrees of knowledge.

## 4. Conclusion

In the present study, we have extended our legacy system to an adaptive learning one and using it we propose a learning model of flipped classroom which has been evaluated through a case study in a C programming class. The results obtained from this case study indicate that the learning model using CBT give a positive contribution to the improvement of the learners' degrees of knowledge.

## Acknowledgements

This work was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) Number JP17K00492 and (B) and Number JP16H03065.

## Reference

[1] Takano Y., Miura K., Tsuji K., Kaneko D., Yamakawa H. & Komatsugawa H. (2014). Development of a Learning Support System for Visualization and Acquisition of Knowledge through Collaborative Problem Posing. Work-in-Progress Poster, The 19th International Conference on Computers in Education. Asia-Pacific Society for Computers in Education 22-24.

# 適応型システムを用いた反転学習モデルの一提案

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## １．概要

能動的学習では，領域知識を踏まえた上での実践が重要で，反転学習を伴う授業設計はこの点で期待が大きい。一方反転学習では，学習者の予習度合いに応じて，授業の理解度・参加度も変わるため，その運営方法が鍵となる。また，高校までの既習知識を前提とする大学初年次系の科目群では，多様な学力分布が影響し，知識の事前修得に対する取扱いが重要となる。

上記の問題を背景に，本研究では修得すべき（又は事前にしておくべき）領域知識に属する問題を Computer-based Testing（CBTe）　and Training CBTr)で反転学習させ，学習者のレベルを授業開始前に可視化した上で課題学習を行う授業デザインを提案する。

## ２．想定する授業内容

本研究では，知識定着からその活用に至る一連の学習能力（以下コンピテンシー）が，授業の複数回分で達成されるような状況を扱う。一つのコンピテンシーの養成には，通常複数の知識セットが内包される。従来の授業では毎回の授業でこの知識群を一定程度の単位で区切り，段階的に教授するのが一般的である。例えば数学の「微分法を活用できる」というコンピテンシーでは，(1)微分の定義（授業第 1 回），(2)初等関数の微分公式（第 2 回），(3)それらを組み合わせた合成関数の微分方法（第 3 回）と段階的に知識が増える。

一方，本研究では一つの到達すべきコンピテンシーとこれを達成するために必要な知識群を一体的に学習させるための授業セット（複数回の授業で構成）を設定する。授業セットを通じてコンピテンシー養成を図るためには，複数知識群の定着・活用・展開が連続的に求められる。これを実現するために，CBT の活用を試みる。具体的には，授業外学習用の演習問題をレベル別に用意する。

問題の難易度の目安は，レベル 1〜2 は知識の定着（言葉の理解・定義の理解）で，レベル 3〜5 が知識の簡単な活用（知識を説明できる・問題を解ける等）で，レベル 6〜7 が知識の発展的な活用（文章題，授業の最終回で提示する課題程度）としている。

学習者には授業セットが開始される前からコンピテンシー養成に必要なすべての知識群を見せる形で，CBTr 機能を用いて予習させる。そして，複数回で実施される授業開始時に確認テストの形で，CBTe 機能を用いて，自らのレベルの確認をさせる。その下で，課題学習を中心とした授業が展開される。

CBTe や CBTr については，我々が開発した IRT 駆動の e ラーニングシステムを採用した（上野）。適応型テストになっているため，学習者の能力に応じて一人一人異なった問題が出題される。第 1 回目の授業では，全体の知識理解・概念理解を目的とするため，レベル 1〜2 を事前の予習の到達目標とする。学習者は，CBTr の機能を活用して学習を進め，主体的に学ぶ中で高いレベルまで達成する者もいる。第 2 回目の予習で，レベル 3〜5 を目標にした場合，第 1 回目授業で概念理解に苦しんだ学習者は，CBTr ではレベル 1〜2 も問題を推奨され，そこに遡って学習することになる。この点で「リメディアル指向」の授業デザインと称している。第 3 回目の実施も同様である。このように，レベル 7 を授業

外学習の到達目標として，反復的に繰り返す中で，予習と復習を一体的に進めることができる。

## 3. 試行結果

提案の授業デザインを本学の情報システム工学科 2 年のコンピュータプログラミング授業に適用した。講義は，学部 1 年のプログラミング授業の継承となっており，講義前半は，1 年次の復習を兼ねて実施される。

第 1 回目の確認テストでは，レベル 2 をクリアすることを努力目標として課した。その結果，レベ 1〜2 が 20 名，レベル 3〜5 が 32 名，レベル 6 以上が 26 名と，一般の学力試験同様で中位が高くなる傾向となった。これは当該領域が一年次の復習を兼ねている内容のためと考えられる。事実，一年次に全く学習していない知識領域に関する別の授業回では，第 1 回確認テストの分布はレベル 1〜2 が最も高くなった。

第 2 回確認テストでは，レベル 4〜5 を到達目標に学習を促した。第 2 回確認テストの結果から，全体的にレベルが向上し，特にレベル 6 以上の分布が最も高い結果となった。レベル 1〜5 程度は知識定着と基本的な知識活用が CBT を通じた反復的な学習を通じて改善されているためと考えられる。こうした学習成果は，リメディアル教育における反復学習の重要性とも一致する結果であり，本授業デザインがリメディアル教育に有効であることを示唆している。一方で，レベル 1〜2 の学習者も散見される。ただ，本授業デザインでは，こうした学習者に対しては授業中に個別に声をかけられるため，早めの改善が期待できる。事実，本試行授業の第 8 回目の確認テスト（知識群は積み重ね）では，レベル 1 は 0 人，レベル 2 は 5 人まで改善した。

最後に，CBT 導入による授業改善と効果について述べる。CBT 導入の結果，授業中は一切インプット型の講義は無くなったため，授業時間すべてワークシートを用いた課題学習に切り替えることができるようになった。また，一人一人の理解を促すため，最初の 15 分程度は PC 教室で個別に学習をさせ，その後 1 時間程度グループワーク（1 チーム 4 名）を課し，グループで課題を解き，メンバー全員が互いに説明し合えるようになること義務化した。この際，最初の 15 分の個人課題の間に，教師が CBT の結果に基づき，各グループに一人必ず当日のレベルの最も高い学習者を割り当てるようにした。この影響で，相互に教え合う状況がかなり活発化した。授業に関する毎回の振返りシートでも，授業開始前は良く分からなかったがグループワークを通じて理解できたという声が多く，本授業を通じたコンピテンシー養成に大きく寄与したと考えている。

## ４．最後に

本稿では，CBTを活用したリメディアル指向の反転授業のデザイン案とその試行を扱った。CBTは一度の設定で試験の提示等が可能で，かつ受講する学習者は多様であって良く，極めて生産性が高いという点で，ICTの効果を発揮できる。また試行を通じて教育効果も高いことが期待される。本研究は，科研（16H03065及び17K00492）の一環で行われている。

# Designing 1st Year Calculus Courses

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**Abstract:** Designing calculus courses, it is difficult to maintain a consistent format while following a "questioning the world" paradigm. It is therefore critical to set design guidelines that motivate students' autonomy noticing the values of the courses, and to analyze knowledge to be understood and absorbed by them. Then one should prepare teaching materials following the guidelines.

**Keywords:** Calculus, FTC, design, applicationism

## 1. Introduction

Teaching design for calculus courses that follows a "questioning the world" paradigm is less likely to be successful, compared to cases of linear algebra courses. Barquero, Bosch and Gascón (2007) designed a research and study course (RSC) for population problems using mathematical modelling and offered it outside of regular courses. The results pointed out "applicationism" and "ecology" (Barquero, 2011). In Japan, Mizumachi and Yamaguchi etc. (2015) designed a practical calculus course with the theme "The birth of Newtonian dynamics" that followed the "questioning the world" paradigm. The course was offered as an elective course available for all college levels, and thus resulted in students with extremely diverse educational backgrounds having different reasons for taking the course. As the result, approximately one-third of the course attendees were satisfied with the course; however, another one-third were not satisfied with the course because they already knew the course contents prior to taking the courses, and the last one-third were not able to understand the course materials. In 2015, Kawazoe also developed and presented a "modeling course for slime mold." His work was positively received, but no institutions, including the university Kawazoe himself worked in, decided to offer the course. This was because there was no appropriate mathematics course that could target 2nd to 4th year students and the content was considered to be too advanced for the 1st year students. So, applicationism and ecology were obstacles to the research. It is decisively important to avoid those obstacles and design ideal differential and integral calculus courses. In the STEM calculus courses, it is necessary to allow students to review the knowledge and skills from high school and then let them build up various knowledge and skills in specialized areas and specific applications. As for teaching concepts, the course will include integral concepts requiring students to modify their previous understanding and to develop them with such as the Fundamental Theorem of Calculus (FTC). However, new important concepts are limited to concepts such as "power series" and "functional approximation." In order to offer a high level of education that uses mathematical modeling, students need to have advanced knowledge in areas other than mathematics, but this is often considered to be an "extra burden". So the needs of the subject are not suitable for "questioning the world" paradigm.

## 2. Necessities of reconsidering and its direction

The following are the list of problems observed at the standard universities in Japan.

- Even in secondary education, students are not good at forming concepts (Fujimura, 2016).
- Even in secondary education, students lack autonomy and most are passive.
- The mathematicians who are responsible for STEM mathematical education are not focusing on application development and are not meeting special needs.
- Even within universities, the diversity in the level of knowledge among students is too wide, and one curriculum does not fit everyone.

To solve these problems, the following educational reformation direction and guidelines are being considered.

- Spend a sufficient amount of time allowing the students to form concepts. Relate to expressive and intuitive images, use praxeology and basic application exercises to link the concepts with skills and practical applications.
- Lower the demand for logic and structure-oriented thinking, but increase the opportunities for intuitive understanding. To this end, use of ICT tools might also be considered.
- Include many applications. For example, include problems on fundamental mechanics using differential equations and basic modelling.
- Spend a sufficient amount of time letting the students gain necessary skills. To save time, use a flip teaching style. Flip teaching may also solve the problem of diversity among students' level of understanding.
- Include study materials containing the history of mathematics and sciences that are meaningful in mathematical education.

## 3. The ideas for reformation

Organize learning knowledge into four categories: concepts, skills, applications. The most important concepts are "Definition and meaning of differentiation," "Definition and meaning of integration," and "FTC," which should be taught carefully. Power series and function approximation are also important concepts. Concepts should be handled intuitively, visualized, and students should be able to apply them in various areas. Justification of concepts should be simplified and intuitive. Understanding and use of concepts does not occur without the development of skills. Incorporating praxeology approaches can be effective. Building of knowledge has to be done by learners themselves. It is important to promote student autonomy during the process. It is quite important to conduct the class in such a way that the students can find "raisons d'être" of the knowledge." Development of calculation skills in differentiation and integration is also required. It is necessary to organize the list of necessary skills and set their levels. In addition, justification of the skills should be simple and justification itself should be considered to be important knowledge. From the observation of recent students, they need to develop the habit of rechecking their calculation results. In terms of applications, in addition to calculations of area, volume, velocity, maxima and minima, differential equations to explain fundamental mechanics and population problems should also be taught. The distribution function in statistics is also an important topic, and the development of various practical, but basic, exercises have been requested. The following are the practical examples of study materials. They are designed to assist students' epistemological changes. This is a change from the current compromised mathematical education to education that focuses on the acquisition of practical knowledge that is usable in global society.

(1) Make differentiation concepts expressive

- Prepare graphs to show the changing ratio of temperature over time using piecewise linear approximation, etc (see 4.1)

(2) To understand FTC intuitively, apply tools for a method developed by M. Artigue, D. Tall, and M. Isoda et al.

- Prepare a tool to verify $f_n' \to f'$ for an arbitrary chosen smooth function $f$ and its polygonal linear approximation $f_n$.
- Confirm that FTC holds polygonal linear approximation $f_n$.
- Discuss methods to show FTC holds for the function $f$.

(3) Example of application - modelling:

A train runs L km from Station A to Station B. In the graph, the horizontal-axis measures time and the vertical-axis measures the velocity of the train. During departure and arrival times, $0 < t < \tau$ and $T - \tau < t < T$, the train moves with constant acceleration. In the interval $\tau < t < T - \tau$, the train runs at a constant speed V.

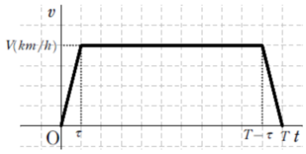


*Figure 1*

(i) Find the speed V , paying the condition that the train reached the station B at the time $t = T$.

(ii) Draw a graph showing the distance from Station A.

(iii) Draw a graph showing the amount of force acting upon the train.

Next, connect suitable quadratic functions to represent the continuous moving velocity during acceleration and deceleration.

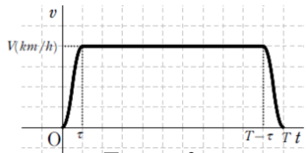


*Figure 2*

(i) Draw a graph showing the distance between the train and the Station A. What is the distance when the train stops at the time T?

(ii) Draw a graph of the acceleration of the train.

The main focus of this question is conceptual understanding. The students are expected to know the integral concepts and FTC and have the skills in regard to function symmetry in order to easily solve the problem. The next section introduces actual teaching materials and plans for a model course (See attached table at the end as reference).

## 4. Actual teaching materials and plans for a model course

### 4.1 <u>Lecture #1: Differential and integral calculus using line graphs</u>

Given a phenomenon of continuous temperature change $f(t)$ (Figure 3), derive a polygonal line approximation to understand the changes in mathematical expression (Table 1). The rate of temperature change represents the velocity of the sun's light energy absorbed/emitted from the air with the phenomena of increasing and decreasing the surrounding temperature. Emphasize that the slope of the line graph shows the rate and draw a graph to show the rate of change (Figure 5).

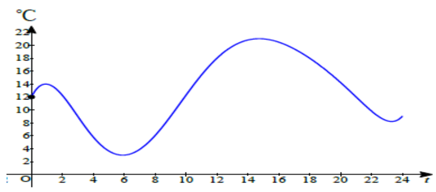


*Figure 3*

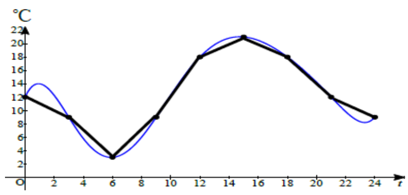


*Figure 4*

*Table 1*

| t(hour) | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| Tem(° C) | 12 | 9 | 3 | 9 | 18 | 21 | 18 | 12 | 9 |

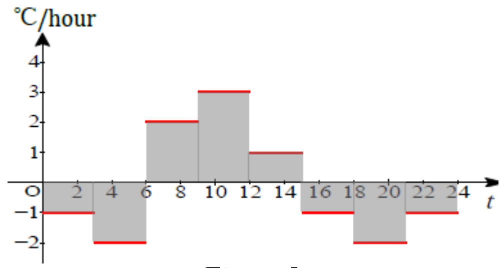

*Figure 5*

In polygonal line approximation, the calculation of the rate of temperature change (conversion from Figure 4 to Figure 5) is "differentiation" and the calculation of the sum of signed area(s) is "integration." FTC definitely establishes polygonal line approximation. Ask students if FTC holds in cases of smooth curves and if so how it can be confirmed.

### 4.2 <u>Leccture #13: Prediction of phenomena from the instantaneous variation rate</u>

Leave a cup containing 40°C warm water in a room kept at 20°C. Measure the temperature of the warm water over time (Figure 6). Let the temperature of the water (°C) be $y(t)$ at a given time $t$ (min). $y'(t)$ is the instantaneous variation rate of the warm water being cooled. Based on the fact that the water cools at a more rapid rate when the temperature difference between the water and the room temperature is larger, hypothesize that $y'(t)$ is directory proportional to $y(t) - 20$. Present this relationship as an equation $y'(t) = k(y(t) - 20)$ to mathematically express the hypothesis. Compare graphs of $y(t)$ that satisfy this equation and the actual phenomena (Figure 7). Emphasize the characteristics of the differential equation for prediction of phenomena

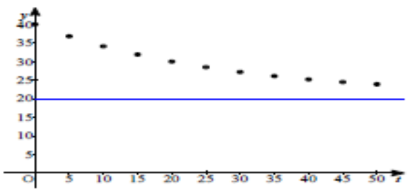
*Figure 6*

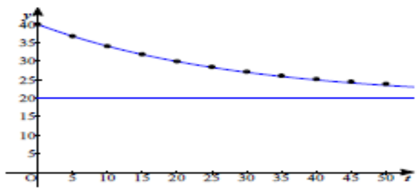
*Figure 7*

### 4.3 <u>Lecture #19: Become familiar with multivariable functions</u>

Understand multivariable functions through actual hands-on activities.

- Consider a situation that can be expressed with two-variable functions (for example, the sum price of two pieces of land that each has different unit price) and provide an actual two-variable function representing the situation.
- Prepare a table like Table 2 and examine the $z$ value at different $x$ and $y$ values. Here, leave some cells open and let the students calculate and fill in the blanks.
- Present 3D graphs of two-variable function with curves cut by several planes and their functions. Let the students understand that they match the values in the table, and ask them to draw some cut curves and write their functions.

Below is an actual example giving a two-variable function $z = 2x + 3y$. Let $f(x, y) = 2x + 3y$. If $(x, y) = (1,2)$, then $z$ can be calculated by $f(1,2) = 2 \cdot 1 + 3 \cdot 2 = 8$. In addition, when cutting the curved surface (or the plane in this case) $z = f(x, y)$ with a plane $y = 1$, the cut curve (or the line in this case) is expressed by $z = f(x, 1) = 2x + 3$. The left graph shows the $z$ value at a given $(x, y)$ of the two-variable function and the right graph is the 3D graph of the function.

*Table 2. Value of the function* $z = 2x + 3y$

| $x$ \ $y$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 6 | 9 | 12 |
| 1 | 2 | 5 | 8 | 11 | 14 |
| 2 | 4 | 7 | | | 16 |
| 3 | | | 12 | 15 | 18 |
| 4 | 8 | 11 | 14 | 17 | |

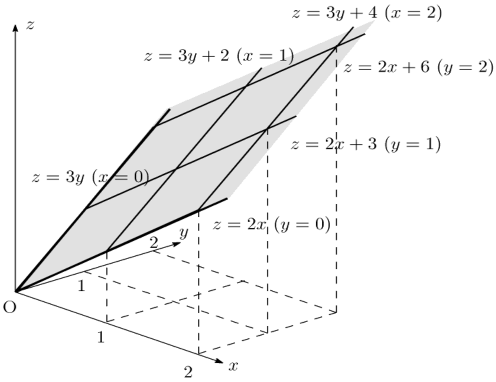


*Figure 8. Graph of the function* $z = 2x + 3y$

Observe sections of this plane $z = f(x, y) = 2x + 3y$ cut by three different planes (Fig. 9). First, $z = 2x$, shown on the left side of Fig. 9, is the graph showing $z = 2x + 3y$ when cut by $y = 0$. This matches the graph appearing on the $y = 0$ plane in Fig. 8. On the other hand, $z = 3y$ in Fig. 9 is the graph showing $z = 2x + 3y$ when cut by $x = 0$. This is consistent with the graph that appears on the $x = 0$ plane in Fig. 8. Then, let the students draw a graph showing $z = 2x + 3y$ when cut by $z = 2$ on the right side of Fig. 9. Express the function as an equation as well.

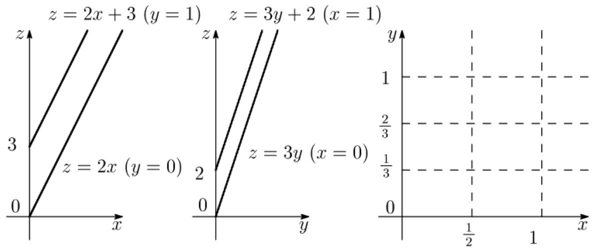


*Figure 9. Sections of the function* $z = 2x + 3y$ *cut by certain plane*

In summary, the reformation of values, reformation based on sufficient analysis of knowledge, and reformation of classes focused on student autonomy are required. It is also important to develop study tools that assist students in self-study so that they are able to retain the knowledge and skills obtained during the class.

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# 初年次微分積分学のデザイン


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微分積分学の場合，「世界探究」パラダイムでのデザインは困難である。結局，「アプリケーショニズム」と「エコロジー」の問題が障害になる。そこで，この科目が持つべき価値と扱う知識を再考・再検討した上で，学生の自律性を引き出し知識を自ら構成できるよう教育デザインのあらましを作り，教材として具体化する必要がある。学生の概念把握力

の弱さや自律性の弱さ，教師の応用への意識の乏しさなど日本の大学で問題になりがちな書点と，諸概念を豊富な内容で構成すること，スキル定着には反転学習が必要であること，微分方程式で力学の基礎を扱うことなど，その対応のために役立つ諸点を挙げ，教材や授業モデルの具体例を示している。最終ページには授業の概要一覧例を挙げた。

*Table 3. Timetable of a Course (Example)*

| | |
|---|---|
| 1. Understand the relationships between differentiation and integration<br>Lecture #1 Differential and integral calculus using line graphs<br>Lecture #2 Use functions to describe phenomenon, understand average variation rate<br>Lecture #3 Understand instantaneous variation rate, differentiation, and derivatives | 2. Understand characteristics of derivatives and perform differentiation<br>Lecture #4 Differentiation of power function, characteristics of derivatives<br>Lecture #5 Differentiation using the chain rule, differentiation of trigonometric functions<br>Lecture #6 Differentiation of exponential/logarithm functions |
| 3. Draw various functional graphs<br>Lecture #7 Derivative test chart<br>Lecture #8 Various functional graphs<br>Lecture #9 L'Hôpital's rule | 4. Understand the meanings and characteristics of integration, obtain calculation skill<br>Lecture #10 Area and integration. FTC. Basic characteristics.<br>Lecture #11 Integration by substitution, integration by parts<br>Lecture #12 Integration of rational function, improper integral(s) |
| 5. Understand the value of differentiation formulas<br>Lecture #13 Prediction of phenomena from the instantaneous variation rate<br>Lecture #14 Equations of motion<br>Lecture #15 Applications | 6. Understand the value of approximation<br>Lecture #16 Power series<br>Lecture #17 Taylor series<br>Lecture #18 Approximation |
| 7. Understand the value of multivariable functions<br>Lecture #19 Multivariable functions<br>Lecture #20 Limit of multivariable functions, partial derivatives<br>Lecture #21 Partial derivatives | 8. Approximation of multivariable functions<br>Lecture #22 Total derivative, composite function<br>Lecture #23 Partial derivative of composite function<br>Lecture #24 Taylor series |
| 9. Application of partial derivatives<br>Lecture #25 Extreme value assessment<br>Lecture #26 Method of Lagrange multipliers<br>Lecture #27 Applications | 10. Multiple integral<br>Lecture #28 Definition of multiple integrals and repeated integrals<br>Lecture #29 Conversion of polar coordinates<br>Lecture #30 Change of variables |

# Designing mathematics education based on the classification of human activities

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**Abstract**: To design a mathematics course/lesson fostering students' ability to use mathematics in real world situations, it is important to grasp the *raison d'être* of the mathematical concepts to be taught from the viewpoint of using mathematics in the real world. In our analysis, human activities in problem-solving situations in which mathematics is used are classified under a three-level structure: social activities, mathematical activities, and pre-mathematical activities. We associate mathematical concepts with five fundamental human activities involving mathematical activities: counting, measuring, comparing, grasping relations, and grasping changes. Moreover, we show an example of the design of mathematics lessons based on the analysis of human activities.

**Keywords**: mathematical literacy education, human activities

## 1. Motivation and background

University mathematics education is divided into three main categories: mathematics education for mathematics majors, mathematics education for non-mathematics majors, and mathematics teacher education. Recently, mathematics education for non-mathematics students has been drawing attention worldwide. In the Japanese context, issues of mathematics education for non-mathematics students are often referred to in connection with mathematical literacy (cf. [6]).

The fact that Japan does not have a successful mathematics curriculum fostering students' abilities to use mathematics in the real world is recognized as a significant problem. The view has often been stated that students at any educational level should be led to develop such an ability. However, as the results of PISA [10] have shown, Japanese secondary school students showed low self-confidence in solving mathematical problems embedded in everyday contexts among the Organisation for Economic Co-operation and Development (OECD) countries. Another survey of university students in Japan also found that university students have a low ability to use mathematics in real-world situations ([12]).

To discuss mathematical literacy education at a university, we have to clarify what mathematical literacy at the university level is. Though there seems no agreement on its definition, here we think of it as the ability to use mathematics in real-world situation as a citizen as well as a professional. More precisely, it is the ability to use mathematics not only in everyday contexts, but also in solving problems in real-world situations requiring higher mathematics: function models using logarithmic/exponential/trigonometric functions, differential equations, multivariate functions, matrix models, Bayesian inference, and other statistical techniques.

How can we foster students' abilities to use mathematics in real-world situations? Sfard [11] pointed out that "our students' ability to summon mathematics when it is most needed will not develop by itself." This suggests that if students learn mathematics only in mathematical contexts, they cannot develop the ability to use mathematics in real-world situations. This leads us to mathematics education using real-world contexts. However, this also has problems.

One of the problems is known as the "situatedness of learning" (cf. [11]). Students often can use mathematics only in the contexts in which they have learned it. Transfer of knowledge is a major challenge for mathematics educators.

Another problem is that when designing a mathematics lesson using a real-world context, teachers tend to design "interesting, but isolated material" ([1]). In [1], Artigue pointed out that the *epistemological analysis* helps us find "*raisons d'être*" of mathematical knowledge, thereby avoiding such a trap.

## 2. Research objectives

In this research, we focus on mathematical literacy education for non-STEM students. In Japan, mathematics education for these students is often discussed from the viewpoint of mathematical literacy.

We think that all mathematics teachers should sincerely ask themselves why they teach mathematics and why students should learn mathematics. If teachers are uncertain of the *raison d'être* of the knowledge to be taught or they have no interest in it, they fall into "monumentalism" (cf. [4]), an educational paradigm of teaching mathematics without conveying why the knowledge to be taught is important. The goal of the lesson, the mathematical concepts taught in the lesson, and the design of the context of the lesson should be backed up with consideration of the *raison d'être* of the taught knowledge. We believe that the analysis of *raison d'être* helps us not only to avoid the trap of designing "interesting but isolated material" but also to overcome the "situatedness of learning." For this reason, we think that analysis of the *raisons d'être* of mathematical concepts is essential to designing mathematical literacy education. In line with the aim of the course that students should develop their ability to use mathematics in real-world situations, we think that it is important to grasp the *raison d'être* of a lesson from the viewpoint of using mathematics in the real world.

In this research, which aims at designing mathematics courses that foster the mathematical literacy of non-STEM students, we focus on the analysis of the *raisons d'être* of mathematical concepts. Our research questions are as follows: (a) What kind of real-world activity is mathematics used in? (b) What kind of mathematical concept is used in each real-world problem-solving situation? (c) For what purpose is each mathematical concept used in each real-world problem-solving situation? (d) How can we design mathematics courses/lessons using the results of these analysis?

## 3. Literature review

There are a number of studies concerning the classification of the real-world contexts in which mathematics is used.

In the 1980s, two important studies were conducted by Mac Lane ([7], [8]) and Bishop ([2], [3]). Mac Lane studied the foundation of mathematics from the viewpoint of human actions as a philosophical study. He listed the human activities from which mathematical concepts originate and also described the connections between the mathematical concepts and human actions explicitly. Bishop regarded mathematics as a cultural phenomenon, and through a cultural anthropological study found six fundamental activities from which mathematics has developed. Bishop also discussed the importance of his results for creating a "culturally-fair mathematics curriculum" ([3]).

Since the beginning of the twenty-first century, such classifications have gained increasing attention within mathematical literacy education. In the assessment framework of PISA [10], the real-world contexts in which mathematics is used, the mathematical content used in real-world situations, and the content topics are classified. Nishimura [9] and Garfunkel et al. [5] classified the objectives of using mathematics in real-world problem-solving situations, aiming to develop principles for designing teaching materials for mathematical modelling education.

Due to constraints of space, here we only refer to the nine objectives of using mathematics in real-world problem-solving situations presented by Nishimura [9]: finding the optimal state, measuring, assessing, capturing trends, investigating possibilities, clarifying structure, counting, calculating, and designing.

## 4. Classification of human activities under a three-level structure

In this section, we present our analysis of human activities concerning the real-world problem-solving situations in which mathematics is used. This research is a joint work with H. Ochiai and G. Gotoh, and is still ongoing. We follow Nishimura in focusing on the aim of problem solving in the real world. In our analysis, human activities are classified under three levels as follows.

*Table 1. The classification of human activities in real-world problem-solving situations*

| **Level** | **Activity** |
|---|---|
| Social activities | predicting, judging, classifying, controlling, optimizing, making a decision, explaining a mechanism/phenomenon. |
| Mathematical activities | counting, measuring, comparing, grasping relations, grasping changes |
| Pre-mathematical activities | collecting information, organizing information (identifying variables, creating tables, representing the data in the graph), thinking stochastically, hypothesizing |

In our view, problem-solving situations in the real world arise with the recognition of certain social activities. Thereafter social activities require one or more mathematical activities, and certain mathematical concepts are called upon when performing the mathematical activities. Moreover, mathematical activities require pre-mathematical activities in the process of "mathematization". Some readers may notice that activities concerning logic, algebra, geometry, and the use of digital tools are not contained in Table 1. In our opinion, thinking logically and using algebra/geometry/digital tools are the most basic skills needed in every activity. Therefore, we put them outside the above three levels.

Our main interest is to describe the *raisons d'être* of mathematical concepts by connecting each concept with class of mathematical activities. We show our provisional result in Table 2.

*Table 2. Mathematical concepts in mathematical activities and their raisons d'être*

| **Activity** | **Mathematical concept** | ***Raison d'être*** |
|---|---|---|
| Counting | combinatorics<br>sum of sequence | a tool for enumerating<br>a tool for accumulating a discrete change |
| Measuring | approximation<br>integration<br>probability, proportion | a tool for estimating quantity<br>a tool for accumulating a continuous change<br>a tool for quantifying possibilities |
| Comparing | statistics<br>derivation | a tool for testing the difference among data sets<br>a tool for finding a minimal/maximal value |
| Grasping changes | number sequence<br>function<br>vectors and matrices | a tool for modelling discrete changes<br>a tool for modelling continuous changes<br>a tool for modelling changes of multi-variables |
| Grasping relations | one variable function<br>multi-variables function<br>vectors and matrices | a tool for representing a relation between two variables<br>a tool for representing a relation of one to multi-variables<br>a tool for representing a relation among multi-variables |

We should remark that the mathematical activity associated with one mathematical concept is not unique because the *raison d'être* of a mathematical concept is multi-dimensional.

Table 2 only gives an overview. Each concept has a substructure and should be described more precisely. The term "number sequence" in "grasping changes" contains the concept of a *recurrence formula* and the concept of the *general term* of a sequence. A recurrence formula is a tool for representing the rule of how to get the next number in a sequence from earlier ones, and the general term is a tool for representing a number sequence as a function. There are many kinds of number sequence, such as arithmetic progressions, geometric progressions, other number sequences defined by $a_{n+1} = f(a_n)$, etc. Each type of number sequence has its own *raison d'être* as a tool for modelling certain discrete change.

## 5. Applications to mathematics literacy education

Now we discuss the use of our classification in mathematics literacy education. We think that the classification presented in the previous section displays the following: the connection between mathematical concepts and their role in the real world, the position of a teaching material in relation to others, and the mutual relations between teaching materials. In this sense, we think our study can help mathematics teachers understand the value of mathematics in extra-mathematical contexts. Moreover, it can help them design mathematics lessons/courses or teaching materials for mathematical literacy education and reflect on their teaching practices. In our opinion, the design of mathematics lessons/courses consists of the following elements: a mathematical concept to be learned, its *raison d'être*, learning contexts, teaching methods, the goal of learning, and assessment materials. In the following, we present an example of the process of designing a mathematics lesson.

**Learning how to use number sequences as modelling tools**: Here we start from a *mathematical concept* to be taught. Let us consider designing a mathematics lesson for students to learn how to use number sequences in real-world situations. First we consider the *raison d'être* of a number sequence. Table 1 shows that a number sequence is a tool for modelling discrete changes. From this, the *goal* of the lesson is determined as becoming able to use number sequences to model discrete changes in real-world situations. In order to determine a *learning context*, we look for examples of real-world situations in which number sequences are used to model discrete changes. For this we take the environmental issue of the significant increase of deer in Japanese forests as an example: there were 3.25 million deer in 2013, and the number of deer is increasing at the rate of 20% per year if left untreated. On this issue, we can treat certain questions in a classroom: (a) How is the number of deer projected to increase in the next decade? (b) How many deer should we catch per year to halve the number of deer within 10 years? (In fact, the Japanese government is trying to halve the number of deer by the end of 2023.) In connection with each question, a number sequence of a different type appears, and we then use this material as the learning context for which we design students' mathematical activities along with these questions for the classroom: students' activities should start with a recognition of the need for "predicting" and "controlling"; then in order to "grasp discrete change," students develop mathematical models for each question through "(pre-)mathematical activities" and analyze each model. Some students, in particular non-STEM students, may need their teachers' help in their learning. Students' activities should be carefully designed along with students' understanding process. In this process, the *teaching method* is developed. Finally, we reflect on the total design of the activities in a lesson and prepare appropriate *assessment materials* (homework, etc.).

We should note that design processes proceed in various ways. One might start from the mathematical concepts to be taught, while another might start from the real-world context in which the mathematics is used.

## 6. Conclusion

In this article, we presented a framework to view mathematics literacy education in the university from the viewpoint of human activities. We believe our framework is helpful not only for university mathematics teachers but also secondary school mathematics teachers in reflecting on their teaching practices in connection with mathematical literacy. However, we note that only showing the classification of human activities (Tables 1 and 2) is insufficient to help novice teachers create new lessons, courses, or teaching materials. In order to help novice teachers, ample examples of sources of teaching materials should be presented in association with the classification tables. Beyond that, we should discuss how to assess teaching practices and students' achievements in mathematics literacy education at the university level. At this point, we plan to write a handbook of instruction and assessment in mathematical literacy education at the university level similar to GAIMME [5].

## Acknowledgments

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# 人間の行為の分類に基づく数学教育のデザイン

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数学を専門としない大学生に対する数学教育の重要性の認識が世界的に高まってきているが，日本では，とくに文系学生を対象とした場合には，数学的リテラシーの観点から言及されることが多い。大学水準の数学的リテラシーの定義について明確な合意はないが，ここではとりあえず，各種関数・行列・微分方程式・多変数関数・確率統計など大学水準の数学までを必要とする現実世界の問題解決場面で数学を活用できる能力（市民としてだけでなく，ある分野を専門的に修めた者としての水準もみたす能力）と定義しておこう。

数学的リテラシー教育では現実世界と結びついた課題を用いた教育が指向されるが，習った文脈でしか知識を活用できない「学習の文脈依存」の問題や，教員が「興味深いが孤立した教材」をデザインしがちであるという問題が指摘されている。これらを乗り越えるためには，M・アルティーグ（[1]）がその重要性を強調しているように，教えようとする知識の「認識論的分析」が重要となる。

本研究では，数学的リテラシーの観点からの数学的知識の認識論的分析として，人間の「行為」に焦点を当てた知識の分類と知識の「存在理由」の分析を試みた。この分析の狙いは，現実的な文脈での数学を用いた課題解決場面での数学的活動の本質を行為とその目的の視点でとらえる枠組みを提供することにより授業や教材の位置付けを明確にし，教員の授業デザインや授業分析などの活動を支援することにある。この分析は，落合，五島との共同研究として進めている最中であるが，現時点での結果を表１に示しておく。（各知識の存在理由の内容や，数学的知識の下部構造の例，また先行研究との関係については英語原稿を参照のこと。）表中，同じ数学的知識が複数の行為と結び付けられているが，これは数学的知識の存在理由が多面的であるためである。

**表1．数学の活用場面での人間の行為に着目した数学的知識の分類**

| 行為のレベル | 行為（数学的知識） |
|---|---|
| 社会的行為 | 予測する，判断する，分類する，制御する，最適化する，意思決定する，仕組みや現象を説明する |
| 数学的行為 | 数える（組合せ，Σ），測る（近似，積分，確率・割合），比較する（統計，微分），変化を捉える（数列，関数，ベクトル・行列），関係を捉える（1変数関数，多変数関数，ベクトル・行列） |
| プレ数学的行為 | 情報を集める，情報を整理する（変数の同定，表の作成，データのグラフ化），確率的に考える，仮説を立てる |

表１に与えたような数学的知識を行為の側面から捉える枠組みは，既存の教材や教育実践の振り返りに有用であると考えられる一方で，新たな授業や教材の開発の支援に役立てるにはまだ不十分な点がある。今後，教材の素材や授業の実践例などを上の枠組みと結び付けて提示するような授業開発ハンドブックの開発などが必要になると考えられる。

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# The analysis and practice of mathematics education based on the concept of affordance

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**Abstract:** Taking mathematical activities as affordances, we propose a rational way of teaching mathematics to non-STEM students. Human activities take place in phenomenal fields and give birth to certain phenomena, which become actualized as affordances. Whether one does or does not become accustomed to mathematical thinking at least partly depends on what phenomenal field is one's usual field of activity. Statistics provides non-STEM students with a phenomenal field in which to find reality in mathematics because its topics and data are full of social meaning. Since statistical thinking is unfamiliar to most students, to classify contents into simple categories is essential to make it easy to follow.

**Keywords:** affordance, mereology, phenomenal fields, statistics

## 1. Introduction

The rational design of mathematics education for non-STEM students needs consideration about how human activities take place. What is responsible for our creative activities? How do they become realized? Although we are accustomed to separate outcomes from the processes through which they are produced, this way of thinking is neither logically necessary nor inevitable. On the contrary, it is quite unique to the Cartesian way of thinking, which presupposes that you can separate the contemplating subject and the external world. It is assumed that you can concentrate your attention on a problem you are addressing without paying any attention to the environment in which it occurs.

While it is the basic structural pattern of scientific experience, it is liable to result in mereological fallacies, that is, fallacies originating from mistaking a whole/part relationship. For instance, you can say that a man thinks, but cannot say that a brain thinks, because a brain is a part of a man and cannot be the subject. That is, 'a brain thinks' is an example of mereological fallacy. [1] Therefore, we need to reconsider our way of thinking about human activities and take them as affordances.

## 2. The concept of affordance

The concept of affordance was first introduced into psychology by Gibson in the 1950s, and has expanded to natural as well as social sciences since the 80s. The basic idea is that you should take environments into consideration when you think about human activities. In this argument the term 'environment' should be taken in the broadest possible sense of the term.

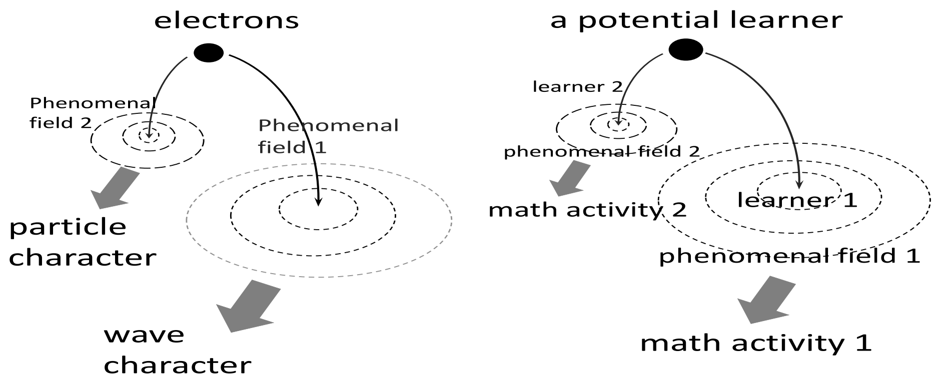
To be more specific in the case of learning mathematics, three components of a hybrid being needed for such a being to be the bearer of an affordance are 1) what is perceived (i.e., mathematical activity), 2) an actor (i.e., a learner of mathematics) and 3) the context in which the perception of an affordance occurs (for instance, the environment in which mathematics is learned). [2] You cannot eliminate either an actor or the context in which a perception of an affordance occurs, though they have not been taken into consideration in scientific thinking. You should place ‘who’ and ‘how’ on an equal footing with ‘what’, instead of just picking out ‘what’. Otherwise, you may easily miss the overall picture of your object and commit mereological fallacies without any consciousness of doing so.

Some of you may recall the Aristotelian concept of first actuality. [3] For instance, the first actuality of an axe is its power to chop wood inasmuch as its constituent matter has been appropriately fashioned into blade and handle. Likewise, any human activity becomes actualized in terms of the triad described above. Therefore, the affordance of a student who is struggling with questions in mathematics in class may not be the same as what is afforded by the same student but in a different environment, say, what is afforded at home.

Human activities become actualized as affordances. That is, human activities take place in phenomenal fields, of which I will explain shortly, and give birth to certain phenomena, which become actualized as affordances. To be addressed in this study is how to analyze mathematics education based on the concept of affordance and suggest how we can improve this education so that it makes much sense to non-STEM students.

As Kant says, we cannot observe physical matter as it might exist in-itself, but only insofar as it appears to us in a certain phenomenon. [4] How does a thing-in-itself become actualized to give a phenomenon? Physical matter falls into a phenomenal field and gives birth to a phenomenon, which will take an appearance characteristic of the field. It is just like a stone thrown into water making waves around it. [5]

Take an electron for example (*Figure 1.* left). When it falls into phenomenal field 1, for example, when you try to see it in an experiment whose set-up is suitable for observing a diffraction phenomenon, you will observe the wave character of the electron as one of its aspects. You may call it affordance number one. On the other hand, when you try to see the electron in an experiment whose set-up is adjusted to generating electron beams, you will observe another aspect of the electron, that is, its particle character. This may be called affordance number two. Thus, you should see the particle-wave character of the electron as a pair of affordances, not as a contradiction.




*Figuare 1. Phenomena, phenomenal fields and affordances*

While an electron is an object whose character is proper to call potentially possible, it becomes a real entity in phenomena because phenomena are actual events which are observable to us. In other words, physical matter, whatever it is, becomes an actual entity as phenomena. As is the case with the electron, it often happens that one and the same object gives birth to various phenomena, each of which is distinct in appearance from the others. The reason why this happens is that it is phenomenal fields that realize phenomena from potentially possible objects. That is, phenomenal fields are conditions, under which phenomena to be observed become realized as affordances in such a way that the phenomena observed are necessary and inevitable to the conditions.

## 3. Phenomenal field and developmental education

We meet our objects to be observed in phenomena and as phenomena as well, because we, too, are parts of phenomena to be realized as affordance. That is to say, the subject as well as its object becomes actualized in a phenomenal field. In other words, the subject itself is nothing but potentially possible unless it becomes actualized in a concrete phenomenon. Thus, a potential learner becomes a real learner, as it were, by taking part in a certain learning activity. (*Figure 1*. right) Whether or not one becomes a good learner in mathematics at least partly depends on the phenomenal field in which one becomes realized as a learner. This is the idea on which to base the design of our developmental education in mathematics. That is to say, students who are poor at mathematics may be in phenomenal fields which are not favorable for learning mathematics. If such is the case, one of the choices available to us is to pull them out of the unfavorable fields, as it were, and to put them in more favorable ones in which they can become realized as strong in mathematics.

For many non-STEM students, the mathematics taught at high school does not seem to have reality. It does not seem to be useful in their present and future lives. In other words, logical consistency or contexts in mathematics may not appeal to their sense of reality. It may be too abstract to follow, full of unfamiliar signs, axioms and theorems, many equations and derivations, and so on and so forth. Thus, it is a subject which does not make sense for ordinary people.

In fact, mathematics is a subject which teaches us how things become clear and distinct by abstracting mathematical structure from seemingly chaotic realities. A scientific theory is a semiotic system. A semiotic triad consists of the sign, its object, and its interpreter. A sign is something which stands for something to somebody in some respect or capacity. The three dimensions of semiotic analysis are the semantic (that is, the relation between signs and objects), the syntactic (that is, the relation between signs and signs), and the pragmatic (that is, the relation between signs and interpreters). [6] Among these relations, mathematics is mostly concerned with the syntactic dimension, which seems intractable for non-STEM students.

To make matters worse there is another element that makes mathematics something intractable for many people. In any science there must exist tacit presuppositions for it to make sense. A distant object looks small, entities with opposite charges pull each other, nature is uniform so that you can assume inductive inferences obtain without exception, and so on. To learn a science is to acquire such presuppositions as well. Non-STEM students who have difficulties in learning mathematics may not have learned such kinds of presuppositions in mathematics in the earlier stages of education. People who do not understand the meaning of a fraction or who cannot transform a ratio into a fraction, may have a hard time understanding the concept of dimension, for instance. They do not know why a unit of density is to be expressed in grams per cubic centimeter.

## 4. The design of mathematics education for non-STEM students

Taking all this into consideration, we think that mathematics education for non-STEM students should be aimed at helping them acquire the tacit presuppositions essential for learning mathematics. It should give proper contexts in which mathematics makes sense for them. If we show them social contexts in which mathematics may serve them well, for instance, they will be willing to tackle it.

Although our college is small with only two faculties and three departments, students are manifold in their interests and majors, and in academic ability as well. Some are interested in computer science, some are studying nutrition, and others are business management majors. Based on these facts we have chosen introductory statistics from a wide range of subjects in mathematics because it is easy to choose social phenomena as topics and present data which have reality for them. Moreover, since it is a brand-new subject for most of them, it is less likely to remind them of unpleasant high school mathematics.

The contents of our introductory statistics course are shown in *Table 1*. Every freshman has to take this course regardless of his or her future major. The contents are classified into four categories of mathematical activities. These are 1) how to count, 2) how to compare, 3) how to estimate, and 4) how to classify. Each of the categories includes well-known mathematical concepts and methods in statistics.

To classify the contents of a course into simple categories makes it easy for students to follow. Rubrics prepared according to a simple categorization principle will help students find which section they are studying and check to what extent they have learned.

*Table 1. The categories of mathematical activities in statistics*

Count: (i) how to count and list data, (ii) how to represent data
Compare: (i) with averages, (ii) with a Z-score and standard distribution
Estimate: (i) interval estimation for population mean, and for population proportion
(ii) scatter plots and correlation coefficients
Classify: (i) menu-matrix analysis

## 5. Examples of questions for introductory statistics

Some examples of topic we use for introduction of our course are listed below.

Question 1 'Is ¥3,500 admission fee to the Observation Gallery of the Tokyo Skytree (450 m) expensive? Answer and explain why.': Students have to look for a rational criterion on which to compare the fee with those for other observatories. By answering the question students are expected to think about the act of comparison, the roles of criteria, how to decide standards, and so on.

Question 2 'What kind of eating and drinking is available at restaurants A to D? (*Table 2*) Choose from a) an organic restaurant, b) a Sushi bar, c) a coffee shop, d) a bar.': This question requires students to think about the implications of figures. Although this question seems to be asking how to compare and decide which is which, we want students to realize, through discussion and research for relevant data, that figures count for much in social contexts.

*Table 2. Figures count much*

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) | (viii) |
|---|---|---|---|---|---|---|---|---|
| A: | 200 | 350 | 100 | 50 | 100 | 800 | 1000 | 200 |
| B: | 1050 | 150 | 300 | 100 | 400 | 2000 | 5000 | 3000 |
| C: | 200 | 500 | 250 | 100 | 100 | 1150 | 3000 | 1850 |
| D: | 400 | 300 | 200 | 100 | 100 | 1100 | 2000 | 900 |

(i) personnel expenses, (ii) raw material costs, (iii) rental expenses, (iv) light, heat water, (v) misc. cost, (vi) total expenses, (vii) earnings, (viii) profits.

Question 3 ‘Classify items in the menu below into three categories A (up to 70-80% cumulative total), B (up to 95%), and C (others), and evaluate the rationality of this classification method.’: This question asks on what basis and to what extent this classification method is rationalized. One possible point for evaluation is whether or not they discover that the data listed in the table are not sufficient for analysis because it lacks prices and benefits.

*Table 3. ABC analysis (simple version without prices and benefits data)*

| Menu | sales (dishes) | % ratio | % cumulative total |
|---|---|---|---|
| 1. Hamburger steak | 3,000 dishes | 30.0 | 30.0 |
| 2. Fried shrimp | 2,500 | | |
| 3. Japanese-style steak | 2,000 | | |
| 4. Omelet | 700 | | |
| ……… | ……… | ……… | ……… |
| 14. Spanish rice dish | 50 | | |
| 15. Lasagna | 30 | | 100.0% |
| Total | 10,000 | | |

Question 4 ‘An elementary school asked 200 pupils whether or not they have breakfast every morning. 168 Pupils answered yes. Estimate the rate of pupils who have breakfast every morning within the 95% confident interval.’: This example asks nutrition students to make the interval estimation for population proportion.

In general, students show interest in these questions and come to do their work willingly. The next task to be addressed is to identify a feasible as well as rational method of measuring changes of ability and attitudes toward mathematics before and after taking our course.

## 6. Conclusion

Firstly, an analysis of mathematics education based on the concept of affordance reveals that contexts in which mathematical problems are dealt with are of critical importance for non-STEM students. Statistics is a possible candidate for introductory mathematics because its topics and data are full of social meaning. Secondly, classifying the contents of a course into simple categories of mathematical activities makes it easy to follow, and consequently, serves to facilitate learning.

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# アフォーダンス概念に基づく文系数学教育の分析と実践

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数学の学習を含む人間のあらゆる活動をアフォーダンスとしてとらえることで，文系学生に適した数学教育のあり方を提案する．人間の活動は現象場のなかで起こり，観察可能な現象を生じる．現象場は，活動主体や，主体を取り巻く環境や，活動の対象によって，それらの相互作用の総体として存在する，と考える．アフォーダンスを考えるときは，活動の主体と対象，およびそれらを取り巻く環境の３者を１つのものとしてとらえることが重要である．このうちのいずれか１つでも欠けると，メレオロジカルフォラシー＝全体と部分の取り違え，が起こる．現象場は本質的に潜在的なものであるが，特定の主体が具体的な対象や環境を選ぶとき，観察可能な現象がアフォーダンスとして実現する．たとえば電子線を発生させるのに適した実験装置を組めば，電子は粒子としての性質を現し，回折現象を観察するのに適した実験装置を組めば電子は波として現れる．粒子と波は相容れないものなので，これらは矛盾した結果のように見えるが，相補的なアフォーダンスと考えれば理解しやすい．もし文系学生の目から見て，数学が現実的で，わかりやすい意味をもつような現象場が存在すれば，彼らにとって数学の学習は無味乾燥な単なる手続きにはならないであろう．数学的な概念や手法を社会的な文脈との関連で考えることは，文系の学生にとって数学の学習を意味のあるものにするために重要なことである．統計学は社会現象と直接的なつながりをもったデータを扱うので，この条件を満たしている．また高等学校までの教育課程では統計学は必ずしも十分に教えられないので，彼らの目に新鮮に映る上に，社会に出てから統計の知識が必要とされる場面は多く，統計学を学ぶことには実際上の利点が大きい．このような観点からわれわれは新入生全員を対象として，可能な限り社会的文脈に配慮して，数的処理という科目名で統計学の基礎を学ばせている．数学的な考え方や態度を身につけさせるのが目的なので，統計実務の専門的な技術を教えることよりも，数える，比べる，推測する，分類する，といった数学の基本的な認識フレームのなかで統計学的概念や手法をとらえることを重視している．このような授業デザインには単純なＩＣＥモデル型ルーブリックが有効である．具体的には，東京スカイツリーの展望回廊への入場料が高いか安いかを考えさせたり，飲食店の経費や売り上げ高のバランスから，それがどのような種類の飲食店なのかを想像させたり，ＡＢＣ分析を行ってメニューを整理する方法の妥当性を考えさせたりといった話題を通じて，数字のもつ社会的な意味を想像させている．こうした訓練は数学的態度を養う一助になるであろう．

**キーワード：**アフォーダンス概念，メレオロジー，現象場，統計学

# Understanding mathematical structures through problem-solving

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**Abstract**: In mathematics education mainly for non-mathematics major students, literacy education that aiming to be able to express mathematically about daily problems including social life and everyday life, and to detect and solve mathematically the essence and structure by use of mathematical thinking is thought to be important. Based on our classification of human activities, grasping daily problems mathematically and rethinking the meaning of using mathematics in everyday situations indicate to compare pre-mathematical activities and mathematical activities and to create meaning the relationship between them. Considering some practical examples of problem-solving classes where going back and forth between social contexts and mathematical contexts, grasping the meaning of solving mathematically and embodying mathematical concepts, it can be expected to foster the ability not only to read and write mathematical expressions and utilize them, but also to see the essence and structure, and utilize mathematics in authentic situations.

**Keywords**: sense-making, reification, contexts

## 1. Introduction

The "advanced mathematical literacy" education we advocate should be concrete which can be realized in university mathematics education. In the first place, mathematics is an academic field from ancient times, and a basic language of various sciences, "citizen's mathematics" as a means of communication and problem solving ([2]). And it is also words to consider quantitatively, to extract essence and to think logically ([1]).

Given this fact, by learning mathematics, it is expected to cultivate the ability and attitudes to capture the essence of the problem, express it in a quantitative (mathematical) way, think logically and solve (including interpretation and examination). Especially, the learning history of students not specialized in mathematics is diverse, and it is not so adequate to place importance on mastery of knowledge and strategies under uniform premise. Rather, education for "using elementary mathematics in a sophisticated form" ([3]), i.e., education that emphasizes nurture of the ability to see mathematical structure by thinking mathematically using relatively basic mathematics in non-mathematical contexts is desirable.

In this sense, education that aims to be able to express mathematically (using diagrams, tables, graphs, formulas, etc.) about daily problems including social life and everyday life, and to detect and solve mathematically the essence and structure by using mathematical thinking is important. In other words, the goals are to grasp the essence of the problem about natural phenomena and social phenomena around us, to express it mathematically (quantitatively), think mathematically (deduction, induction, analogy, generalization, etc.), solve the problem, and to recognize the usefulness and "raison d'etre" of mathematics. These kinds of education can be one of the important points of mathematics education in university as well as in high school.

## 2. Lesson Design

Then, how do we design and practice such education? The purpose of the lesson introduced here is to foster the ability to deeply understand and apply mathematical concepts appropriately. These lesson put emphasis on activities to grasp the overall structure and mathematical concepts of the problem by examining of the way of being applied of mathematical concepts included in the problem. In other words, the point of these practice is to incorporate sense-making activities.

Through a variety of mathematical expressions, by presenting such problem that the meaning of mathematical activities can be conceivable, transformation of expression modes (formula ⇔ diagram, formula ⇔ language, chart ⇔ language, etc.), that is, mathematical and everyday language connection is planned. And that becomes possible to deepen the understanding of mathematical meaning of solution and procedure, and mathematical concepts used there. Alternatively, by arranging problems that have various solutions and can be interpreted diversely through comparative examination of them, social contexts and mathematical contexts are to come-and-go, and pre-mathematical activities and mathematical activities are compared, the relationship between these activities can be focused.

According to our framework, the purpose of solving problems (motivation to use mathematics) is such social activities like "explaining a mechanism/phenomenon" "predicting" "making a decision", mathematics such as simultaneous equations and conditional probabilities are used in such mathematical activities like "grasping changes" "measuring". However, in the process of solving real world problem or unfamiliar (non-routine) problem, while trying and erroring, pre-mathematical activities such as "organizing information" "extracting quantitative relationship out of context" "reading and writing figure/table/graphs" are accompanied.

If you compare these activities with the (well known) mathematical modeling process, pre-mathematical activities are in the real world, mathematical activities are in mathematical world, and "transition" from pre-mathematical activities to mathematical activities is equivalent to "mathematization" which is said to be the most difficult in the modeling process. However, at the trial and error stage and at that of verifying whether the obtained solution is appropriate in light of the purpose, in order to constantly examine what these mathematical activities mean, we often return to pre-mathematical activities. (i.e., doing "de-mathematization" in the process of modeling.) Comparing and examining among pre-mathematical activities, or pre-mathematical activities and mathematical activities not only familiarize themselves with mathematical procedures but also lead to recognize "raison d'etre" of mathematics necessary for solving problems in the real world.

## Example 1: Recognize the meaning of using mathematics in everyday situations

Problem

There are 30 yen and 50 yen stamps. We bought a total of 25 sheets of these two types of stamps and paid 950 yen. How many pieces of each stamp did we buy?

Question

Let's solve it by using knowledge over junior high school.

"What problem is this problem?"

Then, solve by arithmetic method.

"Solve using basic arithmetical operations and diagrams, fractions, ratios, etc. Characters can be used, but keep to a minimum."

Solution (Simultaneous equations)

Suppose I bought a 30-yen stamp x (pieces) and a 50-yen stamp y (pieces).

From the meaning of the problem,

x + y = 25 ··· ①

30 x + 50 y = 950 ·· ②

From ①, x = 25 – y ··· ① '

When substituting ① 'into ②,

30 (25 - y) + 50 y = 950

750 - 30 y + 50 y = 950

750 - 750 - 30 y + 50 y = 950 - 750

-30 y + 50 y = 200

20y = 200

y = 10

Therefore, from ① '

x = 25 - 10= 15

(Answer) I bought 30 yen stamps 15 pieces and 50 yen stamps 10 pieces.

Solution (Arithmetic solution method)

For example, if you purchase □ 30 yen stamps and △ 50 yen stamps, the sum of the areas of the three colors (orange, yellow, purple) is 950. (Figure 1)

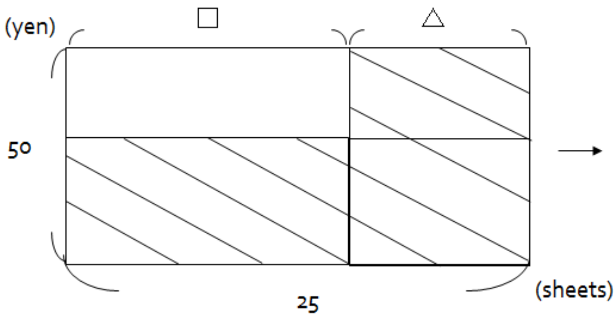


*Figure 1. Area Diagram*

Here, since the area of the orange color is 950 - 30 × 25 = 200,

20 × Δ = 200

Δ = 10 (sheets)

□ = 25 - 10 = 15 (sheets)

Relationship between two solution methods

| < Simultaneous equations > | < Arithmetic solution > |
|---|---|
| 30 (25 - y) + 50 y = 950 | 30 × 25 = 750 |
| 750 - 30 y + 50 y = 950 | (950 - 750) ÷ (50 - 30) = 10 (sheets) |
| 750 - 750 - 30 y + 50 y = 950 - 750 | ... 50 yen stamp |
| -30 y + 50 y = 200 | 25-10 = 15 (sheets) |
| 20y = 200 | ... 30 yen stamp |
| y = 10 | |

The way of solving by the simultaneous equations is equivalent to the idea of assuming all 25 sheets are purchased with 30 yen stamp, dividing the surplus amount by the amount of difference between 30 yen stamp and 50 yen stamp, and counting up the number of 50 yen stamp actually purchased.

## Example 2: Grasp mathematically the problems of everyday life

Problem

Of women aged 40 years, 1% of those who receive regular medical examinations are having breast cancer. Eighty percent of women with breast cancer are positive in mammography, but 10% of women who are not breast cancers also show positive in mammography.

Question

By the way, a woman belonging to this age group was judged to be positive by mammography at periodical medical examination. What do you think of the possibility that this person is really with breast cancer?

Solution (Conditional probability)

Since the probability that this woman is judged to be positive is the case "she is sick & is the examination is correct", and the case where "she is not sick & the exam is wrong",

$$0.01 \times 0.80 + 0.99 \times 0.1 = 0.107$$

On the other hand, the probability that this woman is judged as positive and actually is suffering from breast cancer is

$$0.01 \times 0.80 = 0.008$$

Therefore,

$$0.008 \div 0.107 \fallingdotseq 0.0748 \quad \rightarrow \text{ approximately } 7.5\ \%$$

Solution (Listing)

Assuming that there are 1000 women undergoing breast cancer examinations.

Among them, the number of people who actually have breast cancer are $1000 \times 0.01 = 10$

Among the 10 people with breast cancer, the number of people who show positive by mammography are $10 \times 0.8 = 8$

Women who are free of breast cancer are $1000\text{-}10 = 990$ people,

Among them, the number of people who showed positive by mammography are $990 \times 0.1 = 99$

*Table 1. Classification and Totaling*

| | suffering from cancer | free of cancer | Total |
|---|---|---|---|
| Positive | 8 | 99 | 107 |
| Negative | 2 | 891 | 893 |
| Total | 10 | 990 | 1000 |

From Table 1,

$$8 \div (8+99) \fallingdotseq 0.075 \quad \rightarrow \text{ approximately } 7.5\ \%$$

(Similar to the method of conditional probability)

Relationship between two solution methods

Let the total number be 1 (divide the numerical value in the table by 1000), it's equal to be the value of probability.

Student’s comments (excerpt)

> I thought that each method was unrelated but I understood that they were related, I felt connection between mathematics and arithmetic.

> There were many ways to solve the problem of simultaneous equations without using letters, there were many methods than using mathematical knowledge. The essence of each solution method is the same, I felt mathematics is things that captured the essence.

> I knew that there were two ways to solve the problem, but I didn't think that there is a relationship between the two, so I felt very fresh. Since I already had enough knowledge to notice the relationship, knowledge and knowledge applied like a puzzle and I realized that these knowledge were going to be systematized.

## 3. Discussion

From a viewpoint of the classification of human activities we are developing, thinking the meaning of using mathematics in everyday situations and mathematically grasping the matters of everyday life means to compare "pre-mathematical activities" and "mathematical activities" and to make a relationship between these two.

By focusing on understanding the meaning of solving mathematically (i.e. sense-making) and considering mathematical concepts encompassed in the problem (i.e. reification of concept), we can make a "round-trip" back and forth between social contexts and mathematical contexts. And what is more, we can expect to develop not only the ability to read and write mathematical expressions, ability to utilize these expressions, but also ability to insight the essence and structure in the problem, and the capability of using mathematics in practical contexts. The reaction of the students indicates that it is effective to present problems that have multiple solutions and that can be interpreted variously through comparative examination of them, and incorporate group work as appropriate.

These lessons, that emphasize the development of thinking skills using such mathematical way of thinking, can be practiced using relatively common mathematical problems. And this kind of efforts may be one of the key points of fundamental education in university as well as in high school that put emphasis on mastery of mathematical knowledge and concepts. In addition, by positioning the mathematical activities to be done in the lesson in our framework and presenting the framework of teaching materials development and that of lesson practice, not only it could become a reference for lesson design but also provide hints of lesson improvement.

## Acknowledgements

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# 問題解決による数学的構造の理解

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数学が専門ではない学生、とりわけ文系の学生を対象とした数学教育では、社会生活や日常生活に関わる身の回りの事象について数学的に（図、表、グラフ、式などを用いて）表現し、数学的思考（演繹、帰納、類推、一般化など）も用いて本質や構造を見抜いたり、数学的に解決したりすることができることをめざした教育、言い換えれば、数学の有用性や存在理由を認識し、身の回りで起きる自然現象や社会現象について、その問題の本質を捉えて数学的（数量的）に表現し、筋道立てて考察し、数学的に解決できることを目的とした教育が重要と考えられる。

たとえば、多様な数学的表現を通して数学的行為の意味が考えられるような問題を用意することにより、表現様式間の変換（式⇔図表、式⇔言葉、図表⇔言葉など)、すなわち、数学言語と日常言語の接続が図られ、解法・手続きの数学的な意味や（そこで用いられる）数学的概念の理解を深めることが可能になる。あるいはまた、複数の解法があり、それらの比較検討を通して多様な解釈が可能な問題を用意することにより、社会的文脈と数学的文脈の往還が図られる。本稿で紹介している事例は、我々が開発した数学的知識の認識論的枠組みに基づけば、「仕組み・現象を説明する」「予測する」「意思決定する」「判断する」といった社会的行為が問題を解く目的（数学を使う動機）となり、「変化を捉える」「測る」といった数学的行為において、連立方程式や条件付き確率などの数学が使われる。しかし、現実場面の問題やなじみのない（非ルーチンな）問題を解決するプロセスでは、試行錯誤する中で、「情報を整理する」「量的関係を文脈から取り出す」「図、表、グラフを読み書きする」といったプレ数学的行為が伴うものである。

数学を用いることの意味を日常的場面で捉え直したり、日常生活の問題を数学的に捉えたりすることは、プレ数学的行為と数学的行為を比較し、両者の関係を意味付けることになる。プレ数学的行為同士、あるいはプレ数学的行為と数学的行為を比較検討することは、数学的表現の読み書き能力や活用能力、現実的課題における数学活用力を育成するだけでなく、現実場面の問題を解決するために必要な数学の存在意義を深く認識できることにもつながると考えられる。受講者の反応をみる限り、解法が複数あり、それらの比較検討を通して多様な解釈が可能な問題を用意/提示し、グループワークを適宜組み入れながら学習させることが有効ではないかということが示唆される。

本稿で紹介したような数学的に解くことの意味を把握したり数学的概念を具象化したりすることに重点を置いた取り組みは、比較的ありふれた数学的な問題を使って実践可能であり、大学の基礎教育における一つの重点になり得る。授業で為されることが期待される数学的行為を我々の認識論的枠組みに位置づけたうえで、教材開発や授業実践の枠組みも作成して提示することにより、この種の授業を設計するうえでの参考になるだけでなく、授業改善のヒントも提供できるのではないかと思われる。

**キーワード：**意味づけ、具象化、文脈

# Poster Presentations

## ポスター発表

# Educational Materials on Basic Partial Derivative Which Appeal to Intuition

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**Abstract**: The authors are to publish the educational materials on the calculus of the multi-variable functions for first course students in science and engineering fields [1]. The materials, different from many common-place textbooks, first of all introduce the topics encountered in engineering and then explain the mathematical aspects. This style aims to make readers smoothly understand mathematical concept by extracting their interests and by offering topics which appeal to intuition. This poster exhibits the part of partial derivatives.

**Keywords**: Educational materials, partial derivatives, familiar examples, intuitive understanding

## 1. Intuitive understanding of partial derivatives by familiar examples

Partial derivatives are taken to functions with many variables. We consider a simple case of this type of function and the derivatives using familiar examples. Let us suppose two alloys of gold A and B which include gold uniformly at weight concentrations of 60% and 80% respectively. When we take $x$ (kg) and $y$ (kg) of alloys A and B respectively, the total weight of gold is given by the following form:

$$z = 0.6x + 0.8y\,. \tag{1}$$

This is a multi-variable function with $x$, $y$ as independent variables. It is found that the increases of gold's weight are 0.6 kg and 0.8 kg when we increase the weights of alloys A and B by 1 kg respectively. Numbers 0.6 and 0.8 in equation (1) are respectively the derivatives of $z$ with respect to $x$ and $y$ with the other fixed.

As in this example, the derivative of any multivariable function with respect to one of the independent variables with others fixed is called the partial derivative. Partial derivative is indicated by the symbol "$\partial$." In the above example, numbers 0.6 and 0.8 are the partial derivatives of $z$ with respect to $x$ and $y$ respectively and the facts are written as:

$$\frac{\partial z}{\partial x} = 0.6 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0.8\,. \tag{2}$$

These equations indicate that partial derivative equals to the increase of the functional value due to the unit amount increase of independent variable in question.

As another example, when we buy several kinds of grain with different Kg bids together, the total price is evaluated by the expression like equation (1). In this case, partial derivative of total price $y$ with respect to $x_i$, weight of grain numbered $i$, gives the Kg bid of this grain.

## 2. Graphical interpretation of partial derivatives

Here we consider a general multivariable function as follows:

$$y = f(x_1, x_2, x_3, \cdots, x_n). \tag{3}$$

In this case, the derivative of $y$ with respect to an arbitrarily selected independent variable $x_s$, $\dfrac{\partial y}{\partial x_s}$, represents the slope of the tangent of a curve drawn as a relation between $y$ and $x_s$ with all other independent variables fixed. Figure 1 shows the concept.

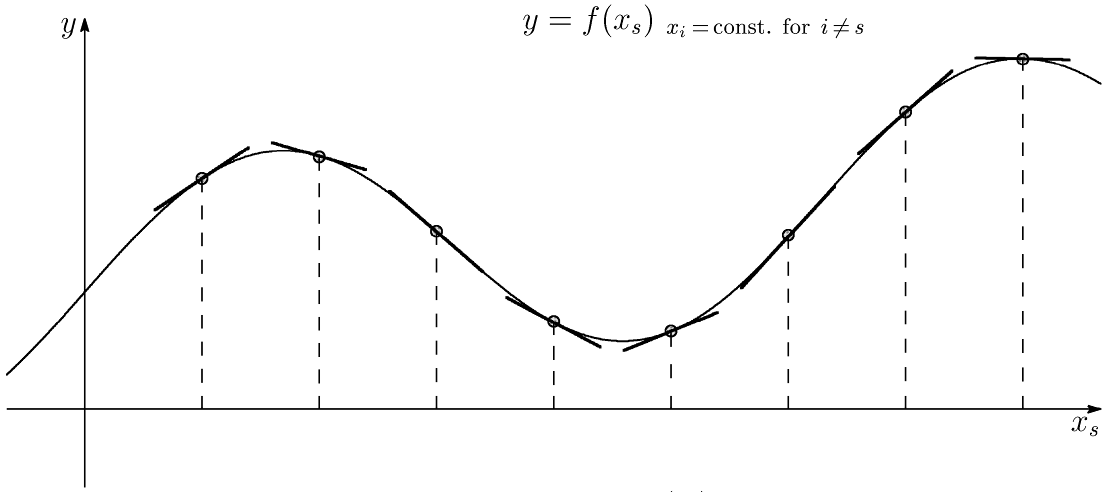
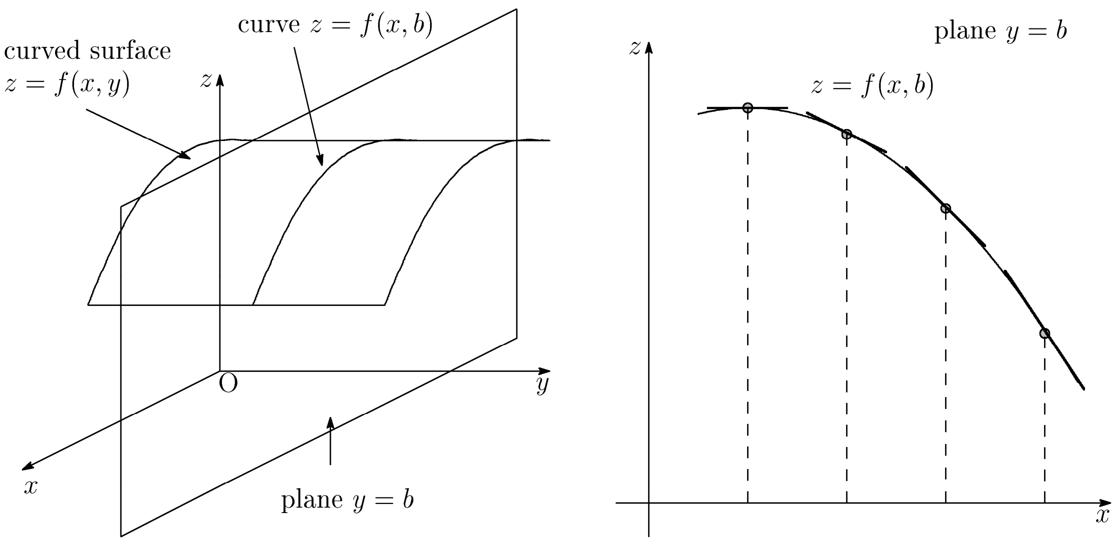



*Figure 1. Conceptual figures of the function* $y = f(x_s)_{x_i=\text{const. for } i\neq s}$ *and the tangents.*




$(a)$ $z = f(x, y)$

$(b)$ *Curve* $z = f(x, b)$ *and* $\left(\dfrac{\partial z}{\partial x}\right)_{y=b}$

*Figure 2. Conceptual figures of function* $z = f(x, y)$ *and* $\dfrac{\partial z}{\partial x}$.

Graphical representation of two-variable function $z = f(x, y)$ is in general given by a curved surface in 3-dimensional space as shown in Figure 2 (a). In this situation, $\left(\frac{\partial z}{\partial x}\right)_{y=b}$ gives the slope of a curve $z = f(x, b)$, which is the intersection of the curved surface given by the function $z = f(x, y)$ and the plane given by the relationship $y = b = \text{const.}$, as shown in Figure 2 (b). Similarly, $\left(\frac{\partial z}{\partial y}\right)_{x=a}$ gives the slope of a curve $z = f(a, y)$, which is the curve of intersection of the curved surface given by the function $z = f(x, y)$ and the plane given by $x = a = \text{const.}$

## 3. Application in science and engineering

In the fields of science and engineering, there are many topics which require the model dealing with the physical quantity as a function of time and space fields. Here we consider an example which does not require us the dimensional consideration. In the evacuation from Tsunami disaster, information of the ground's height is extremely important. At an arbitrary position, we set the direction of the maximum upward slope $I_m$, makes the angle from the north as $\theta$. Also we take the *x*-axis in the east ward, *y*-axis in the north ward, and *z*-axis in the upward vertically. Then slope of the ground in the eastward direction $I_e = \frac{\partial z}{\partial x}$, and the slope in the northward direction $I_n = \frac{\partial z}{\partial y}$ are respectively given as $I_m \sin\theta$ and $I_m \cos\theta$. Therefore, it is clear from the figure in which $I_e$ and $I_n$ are take respectively in the *x* and *y* axes that $I_n \tan\theta = I_e$. Thus the following relation is obtained:

$$\theta = \arctan\left(\frac{I_e}{I_n}\right) = \arctan\left(\frac{\partial z / \partial x}{\partial z / \partial y}\right).$$

This situation is indicated in Figure 3.

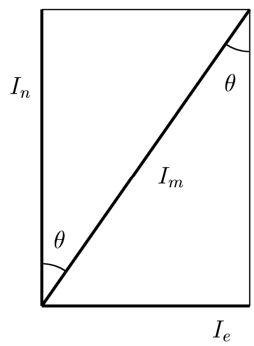



*Figure 3. Relation between $I_m$, $I_e$ and $I_n$.*

**Acknowledgements**

This work was supported by JSPS KAKENHI Grant Number JP16H03065. We would also like to thank the society members of mathematics education related JSPS KAKENHI Grant Number JP16H03065 for giving us lots of valuable comments.

**Reference**

[1] Takagi, S. and Hadano, K. (2017) A textbook of Calculus for multi-valuable functions (in Japanese), in preparation.
[2] Wylie, C. R. and Barrett, L. C. (1995) Advanced Engineering Mathematics 6th Edition, McGraw-Hill, Inc.

# 直感に訴える偏微分基礎の教材

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## １．身近な例による偏微分の直感的理解

金の含有率（重量比）が60％と80％の均一な合金A，Bを想定する．合金A，Bをそれぞれ重量 $x$ (kg), $y$ (kg)だけ取り出したとき，合計の金の含有量 $z$ (kg)は次式で与えられる．

$$z = 0.6x + 0.8y\,. \tag{1}$$

これは $x, y$ を独立変数とする多変数関数である．この式から，合金 A,B をそれぞれ 1kg だけ増やしたときの合計の金の含有量の増加はそれぞれ 0.6 kg および 0.8 kg である．式中の 0.6 と 0.8 という数は, それぞれ $x$ だけを変数 $y$ を定数と考えて $z$ を $x$ で微分したもの, $y$ だけを変数 $x$ を定数と考えて $z$ を $x$ で微分した結果に一致する．このように，任意の多変数関数について，独立変数の 1 つだけを変数と考えて他の独立変数を定数と考えるときの微分を偏微分という．

## ２．図による偏微分の理解

2 変数関数 $z = f(x, y)$ は空間曲面で与えられる．このとき，$\left(\dfrac{\partial z}{\partial x}\right)_{y=b}$ は関数 $z = f(x, y)$ が表わす曲面と $y = b$（定数）が表わす平面との交線 $z = f(x, b)$ の接線の傾きを表す.

## ３．理工学の応用

津波避難行動においては地表の標高の把握が重要である．任意地点で最急勾配 $I_m$ の向きが北向きとなす角度を $\theta$ とするとき，$x$ 軸を東方向，$y$ 軸を北方向，$z$ 軸を標高にとると，東向きの勾配 $I_e = \dfrac{\partial z}{\partial x}$ と北向きの勾配 $I_n = \dfrac{\partial z}{\partial y}$ から $\theta = \arctan\left(\dfrac{\partial z / \partial x}{\partial z / \partial y}\right)$ を得る．

# Teaching Materials for Multivariable Calculus and Vector Analysis Made by KETpic

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**Abstract** : I will report that I made the solid figures to teach students multivariable calculus and vector analysis. To explain the definition of integrals and why we can calculate multiple integrals by iterated ones, I showed many figures by the projectors in classrooms. Some of them are inserted in this article. Regarding the volumes of solid figures as integral values, I could make proofs visible. I used KETpic, which is a macro package for CAS to make TEX files of figure.

**Keywords**: Riemann sum, double integral, surface integral, KETpic

## 1. Introduction

In calculus, derivatives and integrals are defied by taking limits. After explaining the definitions, I show some formulas and teach how to use them to find derivatives and integrals through many examples. After that, many students become able to solve exercises. However, they can do even if they forget how derivatives and integrals are defined. We need not only calculations but also understanding the notions based on limits when we apply calculus to other fields than mathematics. In my class, I show students many figures by projectors in classrooms. I had made image files by graphic tools of computer algebra systems. Three years ago, I begun to use KETpic to make TEX source files of figures which I have inserted to handouts and presentations for my lesson. In my poster, I shall announce my attempts to make materials by KETpic for teaching not only calculus but also vector analysis.

## 2. Double Integrals on Rectangular Regions

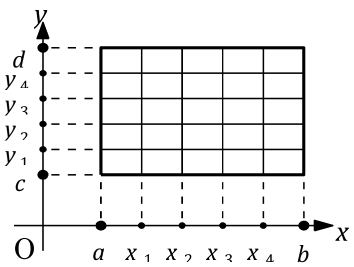

Fig.1

Let $D$ be a rectangular region given by $a \leqq x \leqq b, c \leqq y \leqq d$. Partitioning the interval $[a, b]$ to $n$ subintervals and $[c, d]$ to $m$ ones , I take lines parallel to $x$- and $y$-axes to divide $D$ into small rectangles as Fig.1. The small rectangle $x_{i-1} \leqq x \leqq x_i, y_{j-1} \leqq x \leqq y_j$ is denoted by $D_{ij}$. Choosing a point $(\xi_{ij}, \eta_{ij})$ in $D_{ij}$, the Riemann sum for a function $f(x, y)$ on $D$ is defined by

$$V_{n.m} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) . \qquad (1)$$

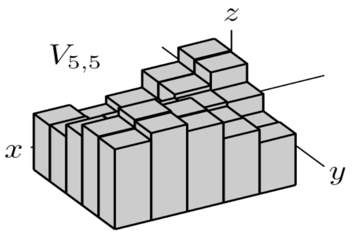

Fig.2

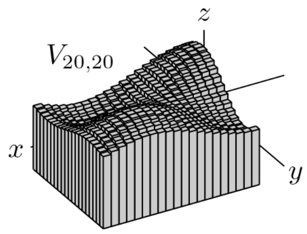

Fig.3

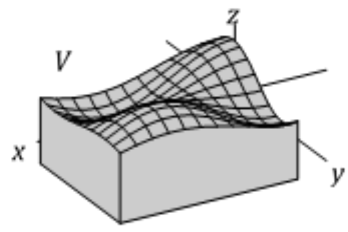

Fig.4

Arranging rectangular prisms whose bases are $D_{ij}$ in Fig. 1 and whose heights are $f(\xi_{ij}, \eta_{ij})$, I can obtain a solid figure whose volume is $V_{n,m}$ like Fig.2. When both of $n$ and $m$ are increasing, we can observe that shading into the solid in Fig. 3, it approaches to the prism bounded below by $D$ and above by the surface $z = f(x, y)$ like Fig. 4.

Increasing $n$ and $m$ to refine the partitions of $[a, b]$ and $[c, d]$, we can show that $V_{n.m}$ approaches to the limit, which is called the double integral on $D$ denoted as

$$\iint_D f(x, y)dxdy. \tag{2}$$

The volume $V$ of the solid in Fig. 4 is equal to (2).

To calculate double integrals, we apply the iterated integral

$$\int_a^b \int_c^d f(x, y)dydx. \tag{3}$$

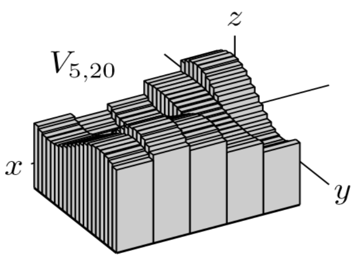

Fig.5

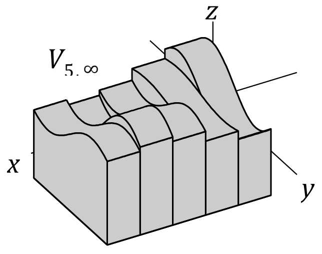

Fig.6

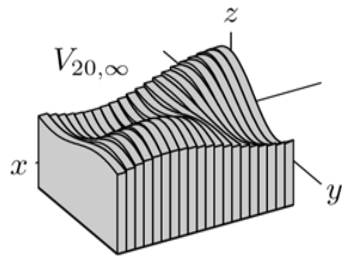

Fig.7

Almost all students can calculate (2) without understanding how to prove that (3) is equal to (2). I tried to show the proof by solid figures to let non-mathematics students understand its outline. I choose $\xi_i$ and $\eta_j$ so that $x_{i-1} \leqq \xi_i \leqq x_i, y_{j-1} \leqq \eta_j \leqq y_j$. Increasing only $m$ in Fig.2, we can see that it turns into Fig.6 through Fig.5. Next, the solid in Fig. 6 become one in Fig. 4 through Fig. 7 when $n$ is increasing.

Because $\sum_{j=1}^{m} f(\xi_i, \eta_j)(y_j - y_{j-1}) \to \int_c^d f(\xi_i, y)dy$ as $m \to \infty$, the volume $V_{n,\infty}$of the solid in Fig. 6 is $\sum_{i=1}^{n} \int_c^d f(\xi_i, y)dy\,(x_i - x_{i-1})$. Since it is the Riemann sum of the function $\int_c^d f(x, y)dy$, $D$ is equal to the iterated integral (3).

## 3. Double Integrals on General Regions

In the case that the region $D$ is not rectangular, I take a rectangle $a \leqq x \leqq b, c \leqq y \leqq d$ so that it includes $D$. Dividing it into small rectangles as Fig.1 and removing ones not lying within $D$ , I can partition $D$ like Fig. 8. Numbering the pieces in Fig. 8 in some order as $D_1, \cdots, D_N$ and choosing a point $(\xi_i, \eta_i)$ in $D_i$, I form the Riemann sum

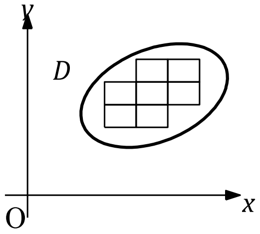

Fig.8

$$V_N = \sum_{i=1}^{N} f(\xi_i, \eta_i) \times (the\ area\ of\ D_i) \tag{4}$$

for a function $f(x,y)$ on $D_i$. Arranging rectangular prisms whose bases are $D_i$ in Fig. 8 and whose heights are $f(\xi_i, \eta_i)$, I obtain a solid figure whose volume is $V_N$ as Fig.9. Refining the mesh in Fig. 8, we can observe that shading into the solid in Fig. 10, it approaches to the prism bounded below by *D* and above by the surface $z = f(x,y)$ as Fig. 11. The limit of (4) is the volume of the solid of Fig. 11, by which we define the double integral (2) on general regions.

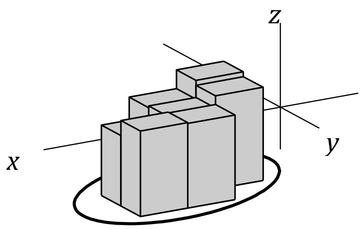

Fig.9

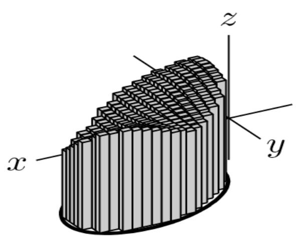

Fig.10

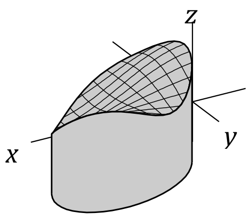

Fig.11

If $D$ is express as the form $a \leqq x \leqq b, \psi(x) \leqq y \leqq \varphi(x)$ like Fig.12, we can calculate (2) by transforming it into the iterated integral

$$\int_a^b \int_{\psi(x)}^{\varphi(x)} f(x,y)dydx. \tag{5}$$

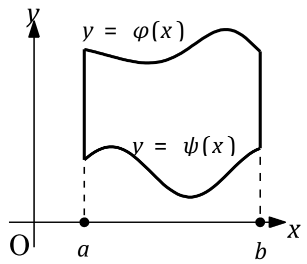

Fig.12

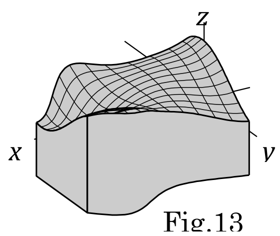

Fig.13

because of what follows. The double integral (2) is the volume of the solid in Fig.13. I partition the interval $[a,b]$ into $n$ subintervals by choosing $n-1$ points satisfying that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and choosing $n$ numbers $\xi_1, \cdots, \xi_n$ satisfying $x_{i-1} \leqq \xi_i \leqq x_i$. Fig.14 is the figure of cross sections of the solid in Fig.13 by the plane $x = \xi_i$. We obtain the arrangement of the thin cylinders whose bottoms are vertical planes in Fig.14 and each thickness of which is $x_i - x_{i-1}$. As the longest length of subintervals $[x_{i-1}, x_i]$ tends to 0, the solid in Fig.15 shades into one in Fig.13 through Fig.16.

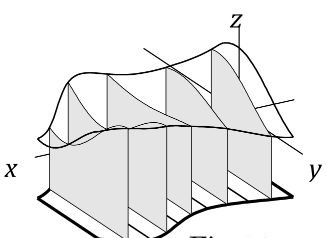

Fig.14

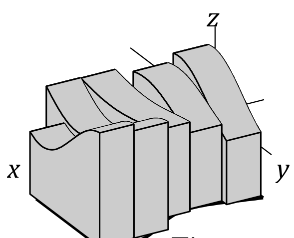

Fig.15

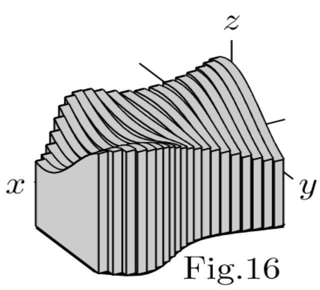

Fig.16

Because the vertical planes in Fig.14 are bounded above by the curve $z = f(\xi_i, y)$ and below by the segment $z = 0, x = \xi_{i,}\ \psi(\xi_i) \leqq y \leqq \varphi(\xi_i)$, their areas are $\int_{\psi(\xi_i)}^{\varphi(\xi_i)} f(\xi_i, y)dy$. Therefore, the volume of solid in Fig.15 or 16 is

$$\sum_{i=1}^{N} \int_{\psi(\xi_i)}^{\varphi(\xi_i)} f(\xi_i, y)dy\,(x_i - x_{i-1}), \qquad (6),$$

which is the Riemann sum of the function $\int_{\psi(x)}^{\varphi(x)} f(x, y)dy$. Because (5) is the limit of (6) , (5) is equal to the double integral (2).

## 4. Changes of Variables and Jacobian

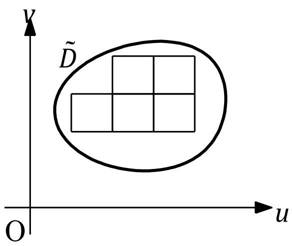

Fig.17

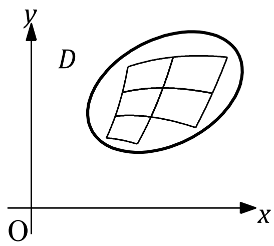

Fig.18

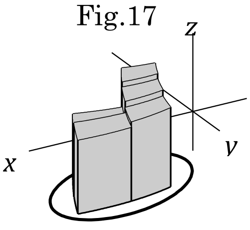

Fig.19

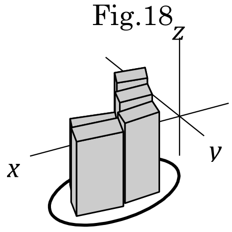

Fig.20

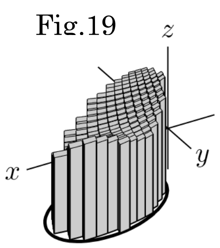

Fig.21

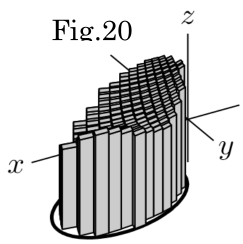

Fig.22

By changing the variables $x, y$ to $u,v$ as $x = \varphi(u, v), y = \psi(u, v)$, we can transform the double integral (2) to one with respect to $u, v$. The region $D$ in Fig 11 is transformed into the region $\tilde{D}$ in Fig.17 by $\varphi$ and $\psi$. I divide $DD$ like Fig. 18 by curves into which we transformed lines parallel to $u$ -and $v$-axes in Fig. 17.

Numbering the pieces in Fig. 18 in some order as $D_1, \cdots, D_N$, and choosing a point $(\xi_i, \eta_i)$ in $D_i$, I obtain a solid figure like Fig.19 consisting of prisms whose bases are $D_i$ and whose heights are $f(\xi_i, \eta_i)$. Its volume can be expressed as (4). However, it is difficult to calculate the areas of pieces in Fig.18 accurately because they are surrounded by curves. Replacing the base $D_i$ by the parallelogram $\widehat{D}_i$, three apexes of which are intersection points of boundaries of $D_{i,}$ I obtain the solid in Fig. 20.

Refining the mesh in Fig. 17, we can observe that the solid in Fig.19 (resp.20) shades into one in Fig.21 (resp. 22). Finally both of the solids in Fig. 19 and 20 approach to the same one of Fig.11. The volumes of the solid in Fig. 19 and 21 is the form

$$\sum_{i=1}^{N} f(\xi_i, \eta_i) \times \left(the\ area\ of\ \widehat{D}_i\right).$$

Appplying the mean-valued theorem, we can prove that the area of $\widehat{D}_i$ is very near to the product of Jacobian and one of the corresponding piece of $\tilde{D}$. Thus we obtain the formula

$$\iint_D f(x, y)dxdy = \iint_{\tilde{D}} f(\varphi(u, v), \psi(u, v))|J(u, v)|dudv,$$

where $J(u, v)$ is the Jacobian of $\varphi$ and $\psi$.

## 5. Surface Integrals

The integrals on surfaces are necessary for vector analysis. Let $S$ be a surface with the parametric form $x = x(s,t), y = y(s,t), z = z(s,t)$. This form can be expressed by the 3D vector function $\boldsymbol{r}(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$. Considering the parameters $(s,t)$ as the coordinate of a point in the plane, we suppose that $(s,t)$ is varying in the rectangular region $D$ like Fig.23. Then, the endpoint of $\boldsymbol{r}(s,t)$ draws the surface $S$ in $\mathbf{R}^3$ like Fig.24.

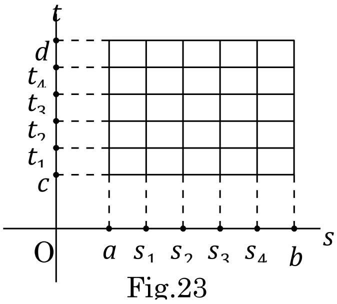

Fig.23

I take lines parallel to $s$-and $t$-axes to divide $D$ into small rectangles as Fig.23. Then, the endpoint of $\boldsymbol{r}(s,t)$ draws the curves dividing $S$ into small pieces as Fig.24 when the point $(s,t)$ moves along these lines. I shall calculate the area of $S$ by summing those of pieces of $S$. Because their boundaries may be not lines, it is difficult to calculate the areas of pieces of $S$ accurately. I replace them by parallelograms, three apexes of which are intersection points of curves on $S$ as Fig 25. It is easy to distinguish between Fig.24 and 25. Refining the mesh in Fig.23, we observe that Fig.25 shades into Fig.26. We look it as a figure that many curves divide $S$ into very small pieces without replacing by parallelograms. By the cross product of 3D vectors, we can calculate the areas of parallelograms. Finally, we can obtain the formula

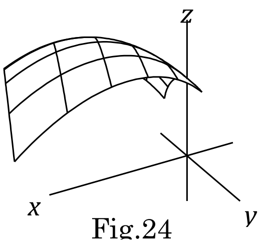

Fig.24

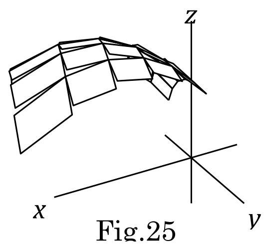

Fig.25

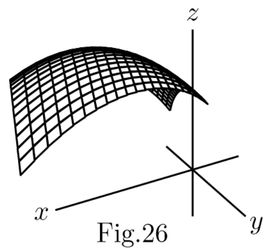

Fig.26

$$\iint_D \left| \frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t} \right| dsdt,$$

which is the area of the surface. We often use approximating small pieces of surfaces by parallelograms to prove some formulas for vector analysis. The figures like Fig.25 and 26 make it easier to understand why cross products are necessary.

## 6.Appendix

I inserted these figures into a calculus textbook written mainly by me. I think that we can find no figures to explain the Riemann sum for double integrals and the equality between double integrals and iterated ones in other textbooks. It is difficult for non-mathematics students to understand the proofs of theorems and formulas. Some of them may calculate derivatives and integrations blindly. I believe that it is a method to make teaching materials as 3D figures by PC.

## Acknowledgements

I would like to thank Prof. Takagi, Prof. Mizumachi, and Prof. Kawazoe for giving me opportunity of a presentation.

# Teaching Materials for Multivariable Calculus and Vector Analysis Made by KETpic

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## １．はじめに

本発表では、発表者が微分積分の授業のために作成した教材の中で立体図形が必要な重積分とベクトル解析に関するものを紹介する。学生に計算方法だけでなく数学的概念を理解させるには式だけでなく図も併用する。立体図形になると作成は難しいが、PCが発達したおかげで、自前で図を作り、授業中に学生に教室に備え付けられているプロジェクターで学生に見せることができるようになってきた。発表者はCASで図のTEXファイルを作成させるためのマクロパッケージであるKETpic ([1])を使ってみた。本発表で披露する図は教科書([2])にも載せたが、他の微分積分の本にはないように思われる。

## ２．重積分

重積分が極限になっているRiemann和が体積になっている立体図形はさまざまな高さの細い直方体を並べたものである。積分領域$D$を座標軸に平行な線分で分割するが、分割を細かくしていくと上が曲面$z=f(x,y)$、下が$D$で挟まれた立体図形に近づくので体積は重積分でるとしてよい。重積分を立体図形の体積に置き換えることにより累次積分が重積分に等しいことの証明を（概略であるが）理解させることが期待できる。特に非数学専攻の学生にとって証明の理解は至難で計算方法の習得のみに陥りやすいので、有効的だと思う。

変数変換については、領域$D$を曲線で分割するので分割された小さい領域の面積の計算が難しい。そこで、3つの頂点を共有している平行四辺形に置き換える。当然、体積も変わってしまうが、分割を細かくすると体積の差がなくなってしまう。この証明は難しいが、図を見せると体積差がなくなっていく様子が視覚的に理解できる。

## ３．面積分

ベクトル解析では曲面上の積分が登場するが、公式の証明だけでなく、公式を実際の現象に適用できるようにするためには、曲面を曲線で分割して考える必要がある。分割された小さい曲面は平行四辺形で近似できるが、図を使って分割を細かくすれば近似による誤差がなくなっていくことを理解させる。また３次元空間における平行四辺形の面積の計算には３次元ベクトルの外積が有効であるので、曲面上の積分の公式に外積が登場する。

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# On a flipped class/learning trial conducted for the linear algebra course.

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**Abstract**: In the second semester of the academic year 2013, the author developed video short lectures for the linear algebra course, which are mainly targeted for engineering school of our university. In 2014 and 2015, the author updated these materials and developed LMS questions, including STACK type questions and true/false questions, associated to the video lectures and tried to conduct a flipped class for non-engineering schools. In the first semester of this academic year, the author developed new video lectures for students to review after class, mainly about foundations of matrices, plane equations, solving linear equations via elementary transformations, calculating determinant, and tried to conduct flipped class/learning. This paper describes the way of these of trials and presents the results of these trials.

**Keywords**: Flipped class/learning, Moodle, STACK

## 1. Introduction

The flipped learning is one of active learning methodologies and usually conducted with video lectures as preparation activities of the students. The linear algebra course is popular and quite important in college mathematics. In Japan, there has been only a few examples of flipped learning/class trials for the linear algebra course.

In the second semester of $2014^{th}$ academic year, the author developed video lectures for students to review after class. Also in the second semester of the following two years, the author taught the students of non-engineering schools and developed more video short lectures for the preparation, all of which are of about 5-10 minutes length. In this academic year, the author tried to conduct flipped learning/class for the linear algebra course of engineering school.

## 2. Video lectures and questions

The video lectures are developed using the software Camtasia Studio[1] and exported to YouTube[2] site and/or local files. The local video files are uploaded to the LMS of our university, which is based on Moodle[3]. The slides used in the video lectures are authored with LaTeX and Beamer class. Moreover, with tikz library, one can give the slides some animation effects which are important for video lectures. In this autumn, the iOS supports the recording of the screenshots of the screen. Using this function, the author made lectures on iPad Pro with an Apple Pencil and encoded them via Camtasia Studio mobile device connection.

As for the questions to check the preparation activity of the students, the author developed more than 100 STACK questions and much more multiple-choice questions and/or true/false questions. Since 2005, in our university, web-based learning system and testing system MATH ON WEB[4], which use *Mathematica* and web*Mathematica,* has been serviced. The site has more than 1000 questions, and so quite useful. However, the system has not linked to our Moodle based LMS, so it is not so easy to monitor and analyze the learning activities of the students. Hence, the author converted the question data of MATH ON WEB to STACK data, and developed more new STACK questions. On the other hand, the author takes note on the usability of smartphones. To answer some STACK questions, the students has sometimes difficulty to input their answers. In order to simply check if the students learned with the videos or not, simple questions like as yes/no questions are useful.

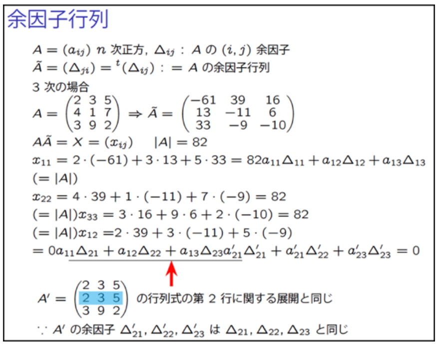


*Figure 1. Slide sample*

## 3. Methods

In this first semester, the author uses the LMS Moodle to manage the learning of the students. The students who learned the video lectures as preparation and tried to pass the "bonus" questions to get extra points, which are additional mark for their score.


*Figure 2. LMS sample shot*

## 4. Results and To-do

The score of my class was somewhat better than the other class of the same school of the college. Actually, the mark of the final exam was 10points greater than the other one in average. At least as for the first semester, where the course curriculum includes only some calculations or procedures of matrices, the author believe that the flipped learning should be effective.

On the contrary, as for the second semester, the course curriculum includes abstract non-sense and the effectiveness is not so clear yet. The author has been mainly engaged in making videos, so it is not quite enough to take care the activities of the students, especially for the second semesters. Thus

It is the next to-do task to develop more effective simple questions and to take care students individually as often as possible.

## Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 15K00926.

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# 線形代数の授業における反転授業／学習の試み

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## １．経緯

2014年度後期工学系の線形代数の授業において，鋭意解説しても聞いていない学生や欠席する学生が多く，対策の必要があった。そこで復習を主目的として，解説動画の開発を開始した。その後，担当の学域・学類が変わり，また線形代数の授業は特に後期において抽象度が上がることから，反転授業を試みることとした。工学域は理系学域として比較的理解力の高い学生がいるが，それでも後期は週2回授業で構成されているのに対し，2014および2015年度の対象クラスは再履修生が多く混在する別学域・学類対象でかつ週1回授業であった。そのため，抽象的な概念は予習が効果的と考えている筆者は，反転授業のように予習活動を中心とする授業時間外学習が重要かつ有効と考えたのである。

## ２．動画教材と確認問題

2014年〜2016年度の動画教材については，スライドをLaTeXとbeamer classで作成し，収録・編集ソフトウェアCamtasia Studio[1]を用いてMacの画面のキャプチャーによる収録で開発した。自習教材としての動画には動きが重要と考えられるので，beamerのプレゼン機能やtikzの描画機能を用い，また，Camtasia Studio の編集機能を使用して動きのある動画教材の開発を試みた。動画は2016年までについてはすべて動画共有サイトYouTubeにアップし，LMSへリンクを掲載した。同時にダウンロードの便宜をはかるため，ローカルファイルサーバーを学内に設置して，学生の利便性に考慮した。

一方，動画の視聴確認に用いる問題としては，紙による指示とアンケート提出の他，本学で供用されているLMS(Moodle)上において，STACKや多肢選択問題，真偽値問題を多数作成した。本研究はスマートホンの活用に注目しており，STACKの問題では入力や表示に難のあるケースがしばしばあり，概念理解や基礎的手順の確認用に入力の簡単な多肢選択問題や真偽値問題(○×問題)も潤沢に開発した。STACKについては，本学では従来，*Mathematica* とweb*Mathematica* ベースのMATH ON WEBを運用しているが，LMSとの連携に難があり，今回はLMS上で完結できるようMoodleの問題タイプの1つであるSTACKへの移植の他，新規開発を行い，実践演習や自宅課題として設置した。

## 3．方法

実際の授業運用としては，学生が予習として動画を試聴すれば簡単に解答できると思われる問題を用意した。また，学生の学習動機を高め予習としての動画視聴を促すことを目的として予習活動に対する課題の評価はボーナスポイントとして与えるようにした。ボーナスポイントはA+評価を与えることを目的として使用した。動画視聴の促進にはゲーミフィケーションが効果的であるというトレント大学のMarco Pollanen氏の主張に着想を得たもので，ゲーミフィケーションではないが，ボーナスポイントの獲得という動機付けを与えようとしたのである。

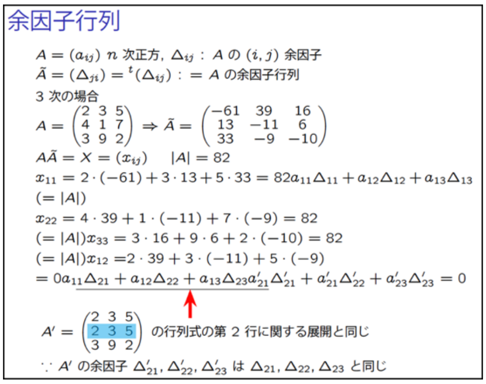


*図 1. スライドサンプル*


*図 2. LMS サンプル*

## 4. 効果と課題

今年度前期において担当した線形代数の対象クラスは1つの学類を2クラスに分割したクラスの1つであるが，期末試験の平均点ベースで約10点程度上であった。これが，反転授業の効果であるかどうかは不明であるが，ある程度の効果はあったと期待される。

ICT活用の問題点はICTを利用しようとしない学生がどうしてもいることを念頭に置いた上で，学生の活動をできる限り注視し，必要に応じて個別の対応をしていくことが重要であると考えられる。これまでは動画開発に注力しすぎたきらいがあるので，今後は学生の理解を確認しやすい問題をより多く開発し， 学生の活動をより細やかに見ていくことができるようにしていく予定である。

### 謝辞

この研究は科研費番号 15K00926 の支援を受けて実施います。

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# A design of introduction to statistics

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**Abstract**: The Japan Statistical Society Certificate (JSSC) Level 2 Examination is a suitable measure by which to evaluate one's level of understanding in regard to statistics knowledge and skills that are taught in general college-level statistics courses. This paper aims to introduce the method taught at the author's college (Japanese standard score rank of about 50) in the "Introduction to Probability and Statistics" course and the evaluation of educational influences employing the JSSC Examination. Among the students who have taken the course, the applicant success rate for the Level 2 JSSC Exam was 78%, while the success rate among all applicants was 44% (7 students passed out of 9 applicants).

**Keywords**: Statistics Education, Japan Statistical Society Certificate Examination, Evaluation of Educational Influences, Freshman Course

## 1. Introduction

Since the early 1990s, a number of countries have worked on systemizing and expanding their statistical education to support the promotion of science and technology. For example, the United States began promoting statistics education with the breakthrough Cobb report in 1992. Currently, statistics education based on the GAISE College Report [1] in 2005 and Common Core State Standards in 2010 [2] is used in elementary-to-university-level introductory statistics courses. This system gives students the opportunity to experience statistical problem-solving from an early age. The *Yutori* education policy (more relaxed education) in Japan has gone against the trends of other countries. In recent years, the course study curriculum for primary and secondary statistics education has been systemized and expanded to improve upon their contents; however, university-level statistics education has not been changed much and the reform of such education is considered to be an urgent project. In 2011, the Japan Statistical Society (JSSC) began to implement the JSSC Examination (level 2) to measure the effectiveness of statistical education taught in universities and to certify one's achievements in undergraduate-level statistics [3]. The JSSC Examination offers 5 levels (1, Pre-1, 2, 3, and 4). Level 2 corresponds to the statistics knowledge and skills taught in university introductory courses. Therefore, this study has revised the curriculum for "Introduction to Probability and Statistics" (half-semester course, 15 classes) and designed the course to improve the level of student understanding so that students are able to pass the Level 2 JSSC Examination at the end of the course.

## 2. Curriculum Contents

The main contents to consider were "Objectives," "Recommendations," and "Effective Textbook Development."

**Objectives:**

Understand the following points by the end of the semester

1. Importance of making decisions based on given data.
2. Information obtained from a random sample of the entire population can represent the population (generalization); however, the selection of a sample influences the reliability of the generalization.
3. There are possibilities that the statistical results may not always match the reality.
4. There are many different statistical methods according to the purpose.
5. To be able to property read and use the regression analysis results. Understand the least squares method.
6. There are many distribution patterns in random variables. The expected value of the random variables is linear.
7. Understand the basics and meaning of binomial distribution, Poisson distribution.
8. Understand the basics and meaning of normal distribution, uniform distribution.
9. Be able to calculate probabilities for each given condition by using appropriate distribution tables among standard normal distribution, student's t-distribution, chi-squared distribution, and F-distribution.
10. Understand interval estimation and deliver an appropriate estimation for the given situation.
11. Understand statistical hypothesis testing and deliver an appropriate hypothesis for the given situation.
12. Be able to apply Bayes' theorem.
13. To be able to read and use the ICT tool analysis results on the multivariate analysis explained in the lecture.

**Recommendations:**

(i) Show the 'big-picture' of the study content's main points and avoid long and unnecessary explanations

Focus on new concepts and statistical ways of thinking and methods. In a context, first explain the basics and general ideas of those concepts, followed by the details. Once the students grasp the 'big-picture' of the contents, they will start drawing a rough sketch of a "knowledge map" and can accept the knowledge that is new to them. Long explanations make concept formation difficult, and prevents students from seeing the whole picture, making learning difficult. Taking a jigsaw puzzle as an example, there are differences in progress between first having a rough image of the whole and then filling in small pieces vs. filling the pieces in little by little without knowing the whole image. It is also recommended to explanations in many contexts.

(ii) Leave the explanation of theories for later and prioritize in-class exercises

Prioritize active learning such as experiments and exercises. Build basic student understanding through the active learning.

(iii) Use calculators for complicated calculations

**Effective Textbook Development:**

Most previously available textbooks followed a style of building up small parts and then gradually increasing understanding of the whole of the study content. To more easily achieve "Recommendations," most pages of the textbook were designed in spread-views, with the big-picture concepts on the left page and the details on the right page.

### 3. Results

Among the students who have taken the course, 9 students sat for JSSC Exam Level 2 in November 2016 and 7 students passed (successful applicant ratio of about 78%). The success rate for all applicants was about 44%.

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# A design of introduction to statistics

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**概要：**

**大学基礎課程で習得すべき統計に関する知識と活用力を評価する試験として“統計検定2級”がある．発表者が勤務校(偏差値50程度)において講義をしている「確率統計基礎」の実践と、統計検定を利用した教育効果の検証について紹介する．統計検定2級試験の全国の合格率が44%のところ、受講生の合格率は78%であった（9名受験し7名合格）**

キーワード：　統計教育、統計検定、教育効果検証、初年次教育

## １．はじめに

2011年から日本統計学会は、大学の統計教育の成果を測り、統計分野の学士力を質的に保証する手段として統計検定（2級）を開始させた。統計検定は1級、準1級、2級、3級、4級があり、2級が大学基礎課程で習得すべき統計に関する知識と活用力を評価する試験となっている。そこで発表者は、勤務校に於いて講義している統計入門科目に値する「確率統計基礎」（半期15コマ）の内容を根本から見直し、受講後には統計検定2級合格に達するレベルになるようデザインした。

## ２．内容

デザインの主な内容は「到達目標」、「推奨事項」、「効果的なテキストの開発」である。

「推奨事項」

(i) コアとなる学習内容の大きな絵をまず示し、冗長な解説は避ける

(ii) 理論の説明は後回しにし、活動的学習を先行させる

実験や演習などの活動的学習を先行させるべきである。活動的学習の中で理論を理解する下地を熟成させる。

(iii) 面倒な計算は計算機で行う

## 3．結果と考察

2016年11月に行われた統計検定2級試験は､受講生は9名受験し7名合格した（合格率約78%、講義以外の講座などの出席者を含むが、その内容からして結果に大きく影響したとは考えにくい)。全国の合格率は約44%であった。教員個人が独自に作成した学習到達目標ではなく、日本統計学会等が定めた水準である統計検定2級試験における受講生の合格率は、自身の統計教育の効果を知るうえで非常に有用であった。

International Workshop on Mathematics Education
for Non-Mathematics Students
Developing Advanced Mathematical Literacy
Tokyo, Japan, January 7-8, 2018

# Use of Microsoft Mathematics in Mathematical Literacy Education

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**Abstract**

Mathematics is not covered in entrance examinations to faculties of humanities and social sciences in Japanese universities, and most students in these faculties in Japanese universities believe they are not very good at mathematics. Because of this, these students cannot be expected to be capable of handling formulas, equations and graphs that form the basis of mathematical literacy, and it is moreover difficult to develop incentives for repetitive practice. This is a report on the exploration of education for mathematical literacy while avoiding repetitive practice through the use of the educational formula manipulation system called Microsoft Mathematics developed by Microsoft. This paper also reports on the knowhow obtained in this process.

Keywords: formula manipulation system, Microsoft Mathematics, mathematical literacy

## 1. Introduction

Mathematics is not covered in entrance examinations to faculties of humanities and social sciences in Japanese universities, and most students in these faculties believe they are not very good at mathematics. And most of them dislike mathematics. Because of this, these students cannot be expected to be capable of handling formulas that form the basis of mathematical literacy, and it is moreover difficult to develop incentives for repetitive practice. One means of dealing with this situation is to avoid dull, repetitive practice by training with the use of a formula manipulation system.

This paper reports on the exploration of education for mathematical literacy while avoiding repetitive practice through the use of the educational formula manipulation system called Microsoft Mathematics ("MS Math" below).

## 2. Using a Formula Manipulation System for Mathematical Literacy Education

While there has been some incorporation of formula manipulation systems into math literacy education in faculties of humanities and social sciences in Japanese universities, such systems are certainly not widely used (For example [1],[2]). One reason for this is the determination by teachers that systems for handling complicated formulas that have been developed for the needs of professional users are ill matched to the education of students who believe they are not good at math.

However, MS Math, a formula manipulation system developed for use in education, is both free and localized into Japanese, and was thus used in classes to develop knowhow related to points that must be kept in mind when using such a system in education, including how best to avoid the aforementioned problems.

## 3. The Formula Manipulation System Used

MS Math[3] is an educational formula manipulation system developed by Microsoft. As of December

12, 2017, the latest version is 4.0 (first released April 1, 2011), and the system is both free and localized into Japanese. It received the Award of Excellence from Tech & Learning Magazine in 2008[4].

The following are three reasons this system was used:

(i) It is free, thus imposing no financial burden on individual students.

(ii) It is a Microsoft product, reducing any mental anxiety students may feel when installing it on individual computers.

(iii) The package was developed for use in education, and its user interface takes into consideration students who may be unfamiliar with computers.

Figure 1 shows the startup screen of MS Math of English version.

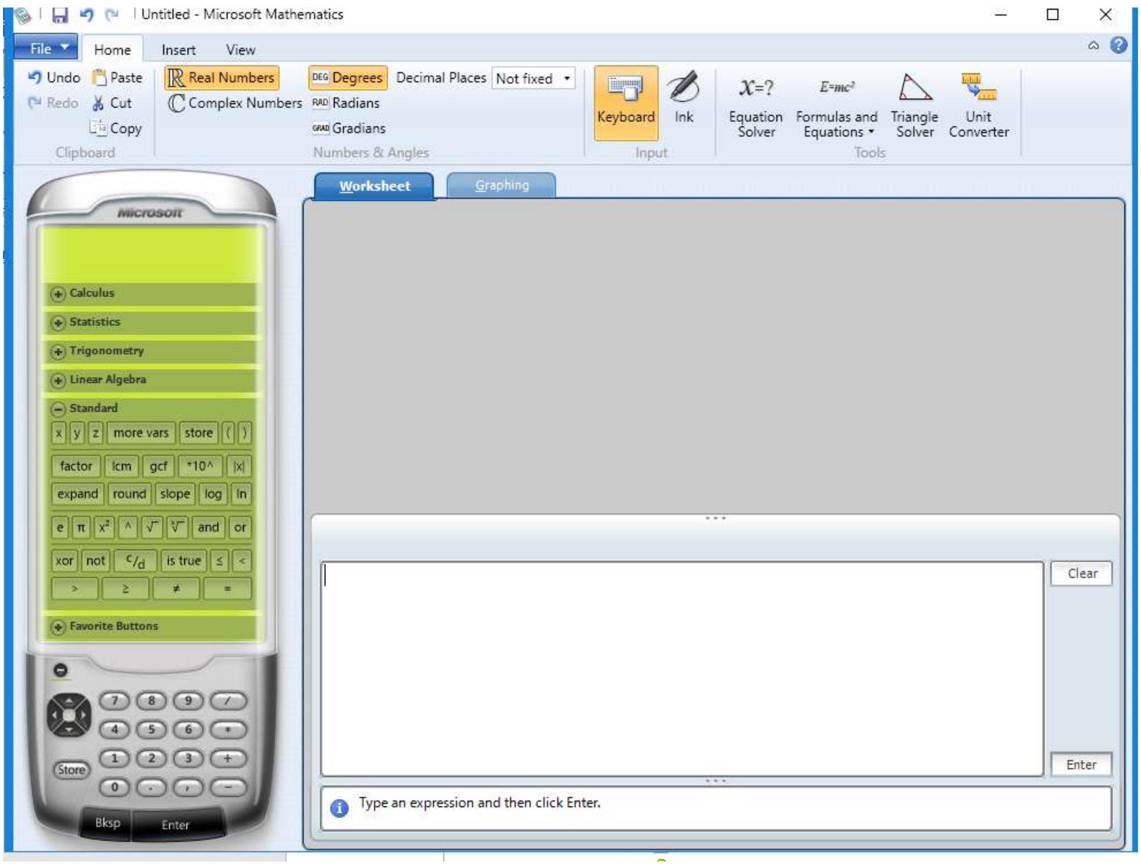

Figure 1. Startup screen of MS Math of English version

As can be seen in Figure 2, buttons are appeared that correspond to typical use for formulas input into the system, and the use of a computer allows even those students that are not very good at math to obtain results.


Figure 2. Appearance of buttons that correspond to typical use

As shown in Figure 3, corresponding solutions are given for equation input by students.

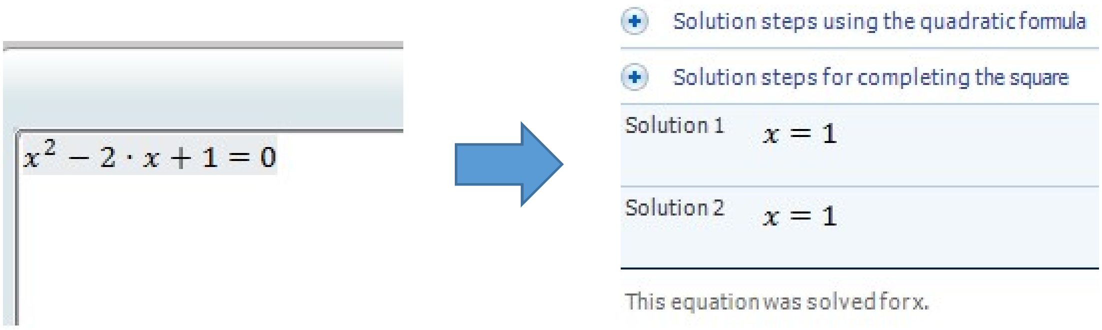


Figure 3. Input and output of equation

Moreover, as shown in Figures 4 and 5, one can draw 2D or 3D graphs with MS Math.

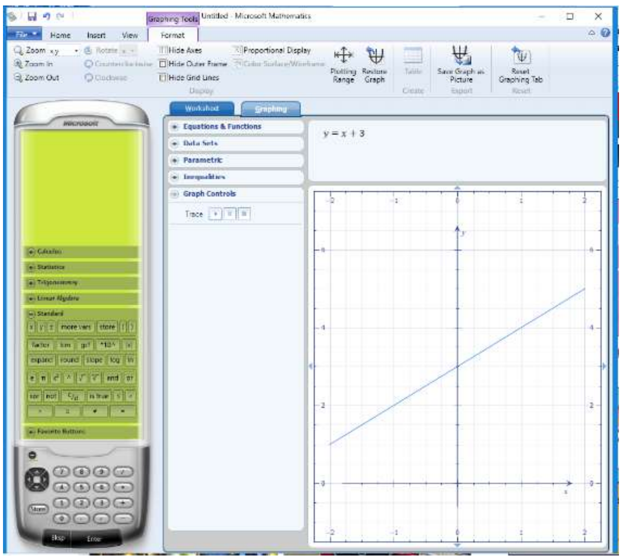


Figure 4. Example of 2D graph ($y=x+3$)

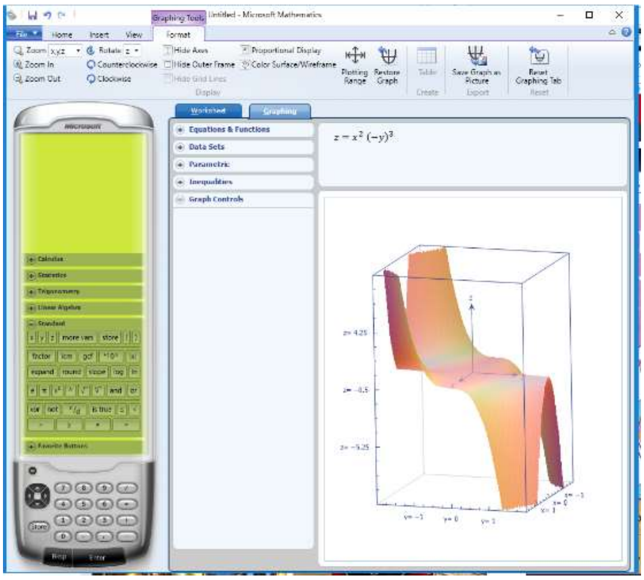

Figure 5. Example of 3D graph ( $z = x^2(-y)^3$ )

## 4. Findings Obtained Through Use

Students reactions showed a positive trend in students' attainment of greater mathematical literacy, as multiple students that had developed an aversion to math in secondary education due to dull, repetitive practice made such comments as, "I wish I had known about this when I was in high school." Such students that had shown a dislike of math, or that harbored a belief that they are bad at math, actually became able to handle formulas that they had given up on.

Below are the main points that must be kept in mind when using MS Math in classrooms, as discovered through use of the system.

First, as shown in Figure 6, the default range of drawing of graphs used in MS Math is not mathematically interesting. This is in contrast to how people draw graphs without a computer, where the characteristic, mathematically interesting points are determined and drawn first. This requires the creation of practice problems taking this fact into consideration.

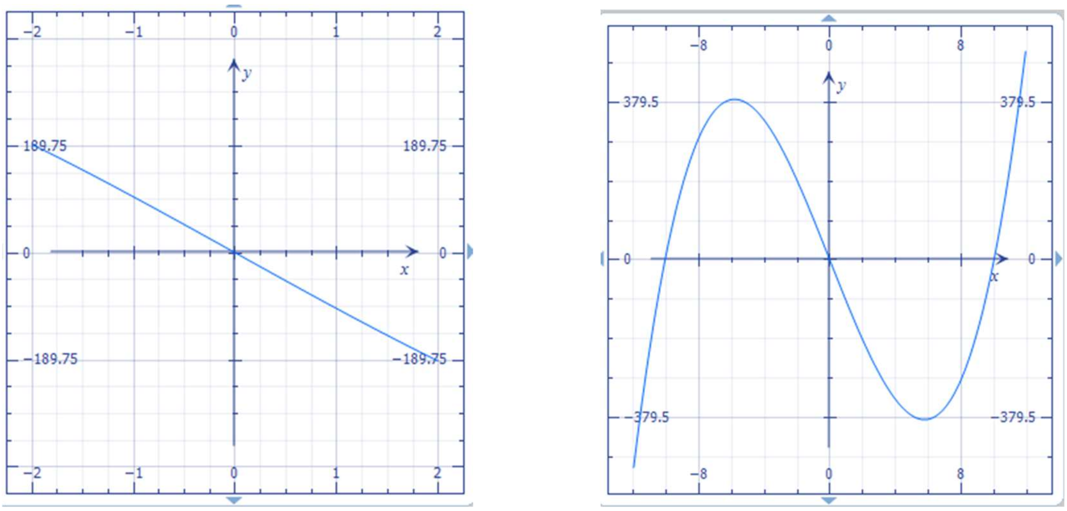


Figure 6. Default and expected graphs of $y = x(x-10)(x+10)$

In addition, when drawing graphs, the horizontal variable is limited to x, and the vertical variable is limited to y. This means that, when drawing the independent and dependent variables that are frequently used in math literacy on a graph, these variables must be changed. This imposes a greater than expected burden upon students, and requires that classes be designed accordingly.

In addition, when inputting as asterisk ("*") to denote multiplication, this is shown on screen as "•" , which confused some students not used to mathematical expressions. Thus, when creating teaching materials for practice, problems must be made with the objective of familiarizing students with automatically generating a clean copy by practicing automatic conversion of these symbols. This automated clean copy includes "*" for multiplication, as well as "^" for exponentiation, and "/" for division.

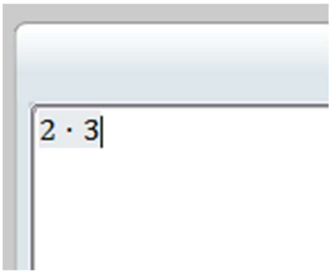


Figure 7. Screen corresponding to input "2*3"

## 5. Summary and Future Topics of Study

Mathematical literacy was taught by using the formula manipulation system MS Math, and by avoiding repetitive practice, which is one factor in students' aversion to math.

This direction showed promise, though at the same time limitations were found in the use of the system in traditional education methods that do not teach formula manipulation. In future school years, we expect to continue to make improvements to self-study materials based on these findings.

Acknowledgements

This research was partially supported by JSPS KAKENHI Grant Number JP 16H03065.

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# 数理リテラシー科目における数式処理システム Microsoft Mathematicsの援用

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## 1. 始めに

日本の大学の人文科学・社会科学の学部では、数学に苦手意識を持っている学生が大半である。そのため、数理リテラシーの基礎となる数式処理能力を期待できず、さらに反復練習へのインセンティブを喚起することも難しい。そこで、教育用数式処理システム Microsoft Mathematics[1](以下 MS Math)に数式の処理をゆだねることによって、反復練習を回避しつつ数理リテラシーを教育する模索について報告する。

## 2.実践で利用した数式処理システムについて

今回採用したMS MathはMicrosoft 社が開発した教育用アプリケーションである。2017年12月12 日現在での最新版はVer. 4.0（2011年4 月1 日提供開始）であり、日本語化もされているフリーソフトウェアである。また、2008年の Award of Excellence from Tech & Learning Magazineを受賞したソフトウェアでもある[2]。

採用理由は、(1) フリーソフトウェアなので、経済的負担が少ない (2) Microsoft社のソフトウェアなので、学生自身のPCへのインストールに抵抗が少ない (3) 教育用に開発されているので苦手意識を持つ学生にも使いやすいという3点である。

## 3. 実践から得られた知見

複数の学生が「高校時代の自分に教えてやりたい」というなど、数学を嫌う学生や苦手意識を持つ学生にとっては、あきらめていた数学を扱えるようになったというところから、更に数理リテラシーについて学ぼうとする良い傾向が見られた。

実践によって判明した、MS Mathを講義で使う際に留意すべき主な点としては：

(1)グラフ描画の際の描画範囲が、デフォルトでは数学的に興味深い範囲とはならない。これは、PCを使わないグラフの描画の際には、特徴的な点を先に決定してから描画するため、意識しなくても数学的に興味深い部分が描画されてしまうことと対照的であり、それを踏まえた演習問題を作成する必要がある。

(2)掛け算を「*」で入力するにも関わらず画面には「・」で現れるなど、キーボードからの入力を自動的に清書した形で表示されるため、数学における表現になれていない学生に混乱が発生した。そこで、演習教材作成時には、このような自動的な清書に慣熟することを目的とした演習問題を作成する必要がある。この様な自動清書には、「*」以外に、べき乗を指示する「^」 や分数を表す「/」がある。

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# The Report of Activities of Academic Support Center in Kogakuin University

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**Abstract**: In Academic Support Center in Kogakuin University, the instructors tutor students in mathematics, physics, chemistry, and English. The students inquire of the instructors about the contents of lectures, homework assignments, examinations, and a review of what students learned in high schools. Through the four subjects, we have a cumulative total of 7,161 students in Tutoring and a cumulative total of 6,464 students in Basic courses in 2016.

In this presentation, we discuss the correlation between frequency of using Academic Support Center for mathematics and students' marks in mathematics in the first quarter and the second quarter of 2017. Two bubble chart will be presented. We concluded that there is a positive correlation between them.

**Keywords**: Tutoring, Basic courses, Academic Support Center

## 1. Introduction

Academic Support Center, or ASC for short, in Kogakuin University was founded in April 2005 for guidance of mathematics, physics, chemistry, and English, which are basic subjects to study their major (i.e. Engineering, Informatics, and Architecture) in the university. There are two types of guidance in ASC:

**Tutoring:** There are students who inquire of the instructors about the contents of the university classes, homework assignments, examinations, and a review of what students learned in high schools. The instructors in ASC tutor to meet such students' needs in a one-to-one lesson or in a group lesson.

**Basic Courses:** The instructors deliver a series of lectures on each subject to the first year students, who want to confirm what they learned in high schools, to brush up their skills or to comprehend the contents of the university classes more deeply.

In a university, students study independently, and such study is different from that in a high school. Usually, a lecture in a university consist of one teacher and many students; therefore, the teacher has some difficulties to pay attention to each student. Also, students cannot ask their questions frankly to the teacher.

Now, there are fourteen instructors in ASC and six of them teach mathematics. Through the four subjects, 7,161 students used Tutoring and 6,464 students attended Basic courses in 2016. According to the questionnaire by ASC in 2014, over eighty percent of the students were satisfied with using ASC.

## 2. Tutoring and Basic courses in mathematics

Kogakuin University has a quarter system, and the first quarter is from April to May, and the second quarter is from June to July. Students have two term-examinations on Calculus in Kogakuin University in the first quarter and the second quarter; Differentials in the first quarter and Integrals in the second quarter. In the first quarter and the second quarter of 2017, we set up 16 Basic courses and we accepted 2,145 students [1]. Actual syllabi of Basic courses of Differentials and Integrals can be found in the two tables below (Table 1 and Table 2). The actual number of students who used Tutoring was 375 (267 of them were in the first year), and a total of 1,945 Tutoring lessons were offered to these students (1,105 of them were for the students in the first year) over the same period [1]. To confirm the effect of activities of ASC, we refer to two bubble charts:

Figure 1 is the bubble chart taking values of;
- the number of Tutoring one student used in the first quarter and the second quarter on the x-axis,
- the score of term-examinations (in the first quarter and the second quarter) on the y-axis, and
- the number of students on those points as the size of the bubbles.

Figure 2 is the bubble chart taking values of;
- the number of Basic courses one student attended in the first quarter and the second quarter on the x-axis,
- the score of term-examinations (in the first quarter and the second quarter) on the y-axis, and
- the number of students on those points as the size of the bubbles.

The results of the first quarter and the second quarter are put together in the bubble charts below.

*Table 1. Syllabus for Basic courses of Differentials*

| Class | Material |
|---|---|
| 1 | Differentiation and chain rule |
| 2 | Derivatives of trigonometric functions and irrational functions |
| 3 | Derivatives of exponential and logarithmic functions |
| 4 | Derivatives of the inverse trigonometric functions |
| 5 | L'Hospital's rule and Taylor’s expansion |

*Table 2. Syllabus for Basic courses of Integrals*

| Class | Material |
|---|---|
| 1 | Indefinite integrals |
| 2 | Integration by substitution |
| 3 | Integration by parts |
| 4 | Definite integrals |
| 5 | Partial fraction decomposition and integrals of rational functions |

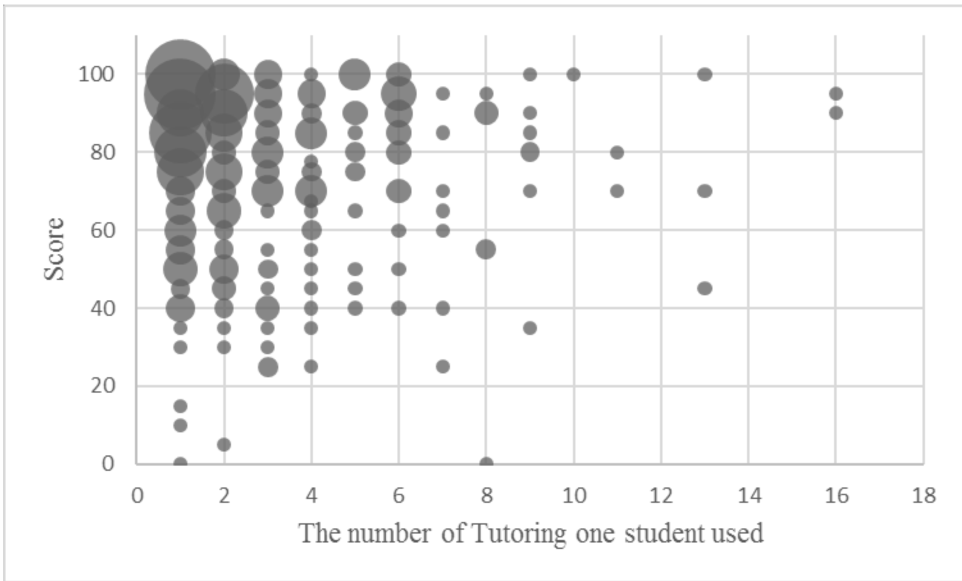


*Figure 1. Tutoring and Score*

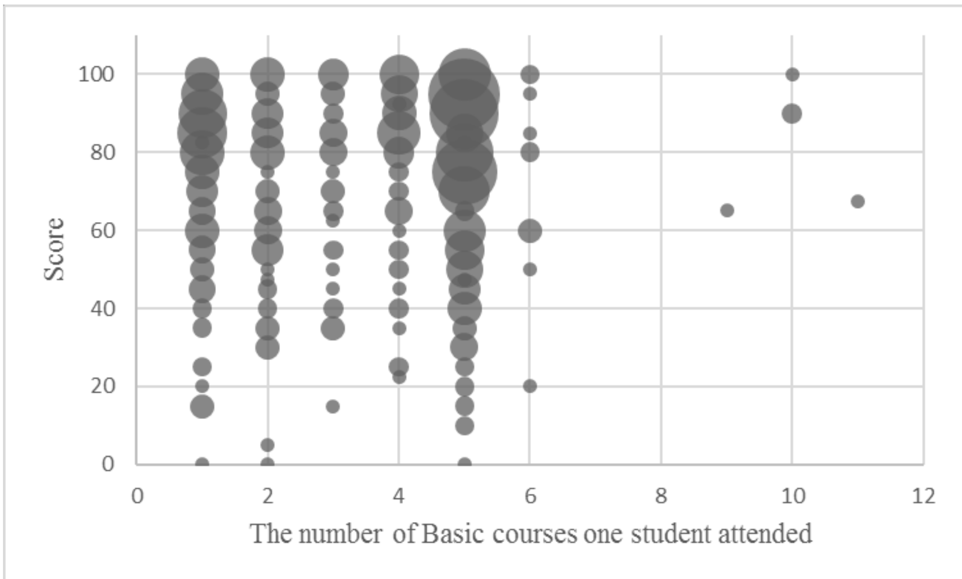


*Figure 2. Basic courses and Score*

One Basic course consists of five classes. There are basically three basic courses which cover the same contents. That is, the students have three or four opportunities to learn the same contents of classes. Therefore, a few students attended more than six times.

## 3. Discussion and Summary

By Figure 1, we have found a positive correlation between earning credits for mathematics and frequency of using Tutoring. In particular, about seventy percent of the students who used Tutoring over six times got passing marks. When students use Tutoring, the instructors can meet each student's needs, including a review of the contents of mathematics in high schools; therefore, it seems that they got such successful results.

By Figure 2, we have observed that many of the students who attended all five classes in Basic courses got good marks in mathematics. However, there were not such results for those who attended the classes only twice, three, or four times. Students who learn in Basic courses every time are earnest, so they study hard in Basic courses, university classes, and at home. Hence, such students can get good marks. However, the students who attended Basic courses only once or twice are not relatively hardworking, so we assume that their study at home would not seem so successful.

We have found that Tutoring and Basic courses help the students to earn passing marks. The instructors can teach each student in Tutoring and many students in Basic courses. Some students want a one-to-one instruction with the instructors, and others prefer to have group lessons. Hence, it is our responsibility to meet the students' needs by offering a different and suitable learning environment. Also, we need to continue our efforts to encourage students to use ASC.

**Acknowledgements**

We would like to express our gratitude to Kiyoshi Kato, Satoru Takagi, and Tomoko Tanaka for helpful advice and useful comments.

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# 工学院大学学習支援センターの活動報告

兒玉浩尚，　野口和範，

工学院大学学習支援センター，　工学院大学学習支援センター，

椎名貴久，　山盛厚伺

工学院大学学習支援センター，　工学院大学学習支援センター

## 1. イントロダクション

工学院大学学習支援センターは2005年4月に設立され，**個別指導**と**基礎講座**の2本立てで学習支援を行っている。2016年度はそれぞれ延べ7,000人の利用があった。

## 2. 個別指導と基礎講座

2017年度第1クオーターと第2クオーターにおける学生たちのセンター利用状況と定期試験成績のデータから，それらの相関を解析した。

## 3. まとめ

個別指導と基礎講座，共に一定の効果があった事がわかった。学生にとって2種類の学習支援が受けられることが効果的であると推察される。

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International Workshop on Mathematics Education
for Non-Mathematics Students
Developing Advanced Mathematical Literacy
Tokyo, Japan, January 7-8, 2018

# Three Wonderful Mathematics Research Papers by High School Students

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**Abstract**: In this paper, we present three wonderful mathematics research papers by high school students who were students of author's class in 2015 and 2017. The titles of their papers are "Pascal zeta functions", " Pythagorean triples on $M_3$" and " ${}^{p}M_n$ perfect numbers", and are all original research works.

**Keywords**: high school students, zeta function, Pythagorean triples, perfect numbers

## 1. Introduction

In this paper, we present three wonderful mathematics research papers by high school students. In mathematics education, to experience mathematics research is very important because students can obtain a lot of knowledge and many skills through this experience. First, students have to find their own problem for their research. Next, they have to gather good data to solve the problem, and they have to extract some laws from the data. They have to make some theorems and have to prove them. If necessary, they have to acquire new knowledge and have to read other papers. Finally, they have to write a paper about their research. The three mathematics research works introduced in this paper are by students of the author's class in 2015 and 2017.

In section 2 of this paper, we treat how to watch for unsolved problems. It is very difficult for students to find their own research theme. Most students don't know what research themes of mathematics are. So, the author makes them watch for unsolved problems as a first step. The important thing is to discuss unsolved problems, not to try to solve them. They learn how to set up their own problems by these discussions. In section 3, we treat how to advise students to decide their research themes. The teacher's advice should help students develop their ideas for their research. The author will introduce two examples.

In section 4, we show the results of the first research whose title is "Pascal zeta functions". Pascal zeta functions was defined by Huga NAKANO, who was a student of author's class in 2015. The first line of Pascal triangle is the sequence $\{1, 1, 1, \cdots\}$. The 2nd line is the sequence $\{1,2,3, \cdots\}$ by natural numbers. The 3rd line is the sequence $\{1, 3,6,10, \cdots\}$ by triangle numbers. Nakano imaged the Riemann Zeta function $\zeta(s) = 1^s + 2^s + 3^s + 4^s + \cdots$ by the sequence of the 2nd line. Moreover, he considered the function $P_3(s) = 1^s + 3^s + 6^s + 10^s + \cdots$ by the sequence of the 3rd line. In general, he defined the $n$-th Pascal zeta function $P_n(s)$ by the sequence of the $n$-th line of Pascal triangle. In section 5, we show the results of the second research whose title is " Pythagorean triples on $M_3$". Here $M_3$ is the subring of the matrix ring with degree 2, i.e.,

$$M_3 = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \text{ are non negative integers.}\right\}$$

This was researched by Yoshifumi YABE who was a student of author's class in 2015, as well. He defined the Pythagorean triple $(A, B, C)$ on $M_3$ by $A^2 + B^2 = C^2$. Furthermore, he asked a question, which is "Can you define primitive Pythagorean triples on $M_3$?" He started his mathematics research to solve this question. In section 6, we find the results of the third research whose title is " ${}^{p}M_n$ perfect numbers". Here $p$ is an odd prime number and ${}^{p}M_n$ is denoted by $1 + p + \cdots + p^{n-1}$ and it is called a $p$- Mersenne number. ${}^{p}M_n$ perfect numbers is defined by Shogo KIRIYAMA who is a member of the Math club and a student of author's class in 2017. A positive integer $a$ is called a perfect number when $a$ is equal to the sum of its proper positive divisors. It is well known theorem that $a$ is an even perfect number if and only if $a = 2^{n-1} \times M_n$ where $M_n = 1 + 2 + \cdots + 2^{n-1}$ and is a prime number. This theorem was found by Euclid and was proved by Euler. So, Kiriyama asked a question which is "Which properties do the numbers $p^{n-1} \times {}^{p}M_n$ have?" He started his mathematical research to solve this question and could define ${}^{p}M_n$ perfect numbers.

## 2. Watch for unsolved problems

To watch for unsolved problems for students is very important. Because it helps students find their research theme. By observing their reaction, teachers can understand their interests. But, the author does not require students to attack unsolved problems because it is very difficult to try to solve them. An important guidance is to make students consider some mathematical ideas by treating unsolved problems. Actions like these lead to determining their research theme.

Example 1 (Riemann hypothesis). The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $1/2$. Furthermore this hypothesis implies results about the distribution of prime numbers.

To consider the Riemann hypothesis, the author showed students the following equalities:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \qquad \zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

$$\zeta(0) = \text{"}1{+}1{+}1{+}1{+}\cdots\text{"} = -\frac{1}{2} \qquad \zeta(-1) = \text{"}1 + 2 + 3 + 4 + \cdots\text{"} = -\frac{1}{12}$$

$$\zeta(-2) = 1{+}2^2{+}3^2{+}4^2{+}\cdots = 0 \qquad \zeta(-3) = \text{"}1 + 2^3 + 3^3 + 4^3 + \cdots\text{"} = \frac{1}{120}$$

Most students can not understand the importance of the Riemann hypothesis. But, most students are surprised by the values of $\zeta(s)$ when $s$ is negative.

*Why is the sum of natural numbers equal to* $-1/12$*?*

*Why is the sum of square of natural numbers equal to zero?*

A student made the following interesting remark.

*The first line of Pascal triangle is the sequence* $\{1, 1, 1, \cdots\}$.
*The 2nd line of it is the sequence* $\{1,2,3,\cdots\}$ *by natural numbers.*
*The 3rd line is of it the sequence* $\{1, 3,6,10, \cdots\}$ *by triangle numbers.*
*Now, we know*

$$\text{"}1{+}1{+}1{+}1{+}\cdots\text{"} = -\frac{1}{2}, \qquad \text{"}1 + 2 + 3 + 4 + \cdots\text{"} = -\frac{1}{12}$$

*So, what is the sum "*$1 + 3 + 6 + 10 + \cdots$*" of triangle numbers?*

Example 2 (Perfect numbers). A perfect number is a positive integer which is equal to the sum of its proper positive divisors. The first perfect number is $6$. Its proper divisors are $1,2$, and $3$, and $1 + 2 + 3 = 6$. The next perfect number is $28 = 1 + 2 + 4 + 7 + 14$. This is followed by the perfect numbers $496$, $8128$, and $33550336$. On the other hand, the following theorem for even perfect numbers is well known.

*Theorem (Euclid-Euler).* $a$ *is an even perfect number if and only if* $a = 2^{n-1} \times M_n$ *where* $M_n = 1 + 2 + \cdots + 2^{n-1}$ *is called Mersenne number.*

The following are unsolved problems.

"*Are there infinitely many perfect numbers?*"

"*Are there any odd perfect numbers?*"

Most students said the answers to these problems will be "yes". Most students said that we needed a supercomputer to research the above problems. A student made an interesting remark.

*How about a prime number* $1 + 3 + \cdots + 3^{n-1}$*? For example, the prime number* $13$ *equals to* $1 + 3 + 3^2$ *and the proper positive divisors of the number* $117 = 3^2 \times 13$ *are* $1, 3, 9, 13, 26$. *But,*

$$117 \neq 1 + 3 + 9 + 13 + 26.$$

*Hence,* $117$ *is not a perfect number.*

## 3. Advice to decide the research theme

The activities of students must be developed by the teacher's advice. What is appropriate advice? It is to present a method for them to get mathematical data. In most mathematics research, mathematical data is obtained by some calculation. So, their mathematical ideas should change into forms that can be calculated.

Example 3 (Advice by using Fibonacci Dirichlet Series). The way of calculation of Fibonacci Dirichlet series by L.Navas[L] is as follows.

*Let $F_n$ be the $n$-th Fibonacci series and let's $\varphi$ be the golden ratio. Then we have the following calculation:*

$$\sum_{n=1}^{\infty} F_n{}^{-s} = 5^{\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^{k(n+1)} \varphi^{-n(2+k)}$$
$$= 5^{\frac{s}{2}} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k \sum_{n=1}^{\infty} (-1)^{k(n+1)} \varphi^{-n(2+k)}$$
$$= 5^{\frac{s}{2}} \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{\varphi^{s+2k} + (-1)^{k+1}}$$

The author advised a student who had an interest in the sum of triangle numbers. In the above calculation, it is important to switch the operations $\Sigma_{n=1}^{\infty}$ and $\Sigma_{k=0}^{\infty}$ in the second equality. He soon studied this way.

Example 4. (Advice for Perfect numbers) In example 2, the author was interested in a student's question "How about a prime number $1 + 3 + \cdots + 3^{n-1}$?". The author advised him with the following.

*When $M = 1 + 3 + \cdots + 3^{n-1}$ is a prime number, I think that to think about the number $3^{n-1} \times M$ is very interesting, too. Are there other characteristics different from perfect numbers in numbers like $3^{n-1} \times M$?*

After this advice, the student found the equality

$$117 = 2 \times (1 + 3 + 9 + 13 + 26) - 13.$$

Moreover, he considered the prime number $1093 = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$ and the number $3^6 \times 1093$. So, he showed the equality

$$3^6 \times 1093 = 2 \times (\text{the sum of proper divisors of } 3^6 \times 1093) - 1093.$$

In general, he proved the rule

$$3^{n-1} \times M_n = 2 \times (\text{the sum of proper divisors of } 3^{n-1} \times M_n) - M_n$$

He was very interested in this rule and he said "*How about a prime number $1 + p + \cdots + p^{n-1}$?*".

## 4. Pascal zeta functions

This research is by H. Nakano (cf. pp149-153 in [1]). Let $\{{}^N a_n\}$ be the sequence of the $N$-th line of Pascal triangle and let $P_N(s) = \sum_{n=1}^{\infty} ({}^N a_n)^{-s}$. Let's see the way of the calculation of $P_2(s)$ by Nakano.

$$({}^2 a_n)^{-s} = \frac{n^{-s}(n+1)^{-s}}{2^{-s}} = 2^s n^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-s-k} = 2^s \sum_{k=0}^{\infty} \binom{-s}{k} n^{-2s-k}.$$

So, we have

$$P_2(s) = \sum_{n=1}^{\infty} ({}^2 a_n)^{-s} = 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-2s-k}.$$

In the above equality, whether or not $\Sigma_{n=1}^{\infty}$ and $\Sigma_{k=0}^{\infty}$ can be switched is a problem. But, he continued the calculation by assuming that this operation was correct. As a result, the equality

$$P_2(s) = 2^s \sum_{k=0}^{\infty} \binom{-s}{k} \zeta(2s+k) \tag{4.1}$$

was obtained. From this, we have

$$\text{“}1 + 3 + 6 + 10 + \cdots\text{''} = P_2(-1) = 2^{-1}\left\{\binom{1}{0}\zeta(-1) + \binom{1}{1}\zeta(0)\right\} = -\frac{1}{24},$$
$$\text{“}1^2 + 3^2 + 6^2 + 10^2 + \cdots\text{''} = P_2(-2) = 2^{-2}\left\{\binom{2}{0}\zeta(-4) + \binom{2}{1}\zeta(-3) + \binom{2}{2}\zeta(-2)\right\} = \frac{1}{240}.$$

After that, he studied the Euler-Maclaurin formula, and so he tried to calculate $P_2(s)$ by using the formula. Also he had $P_2(-1) = -1/24$ and $P_2(-2) = 1/240$. He and the author think that above formula (4.1) gotten by him is correct.

In general, he considered the formula of $P_N(s)$. In particular, he had

$$P_N(-1) = -\frac{1}{N!}\sum_{k=0}^{\infty}\begin{bmatrix} N \\ k \end{bmatrix}\zeta(-k)$$

where $\begin{bmatrix} N \\ k \end{bmatrix}$ is the Stirling number of the first kind by $N$ and $k$. He likes the above expression very much. Moreover, he studied an integral expression of $P_2(s)$. So he proved the following formula

$$P_2(s) = \frac{1}{2\Gamma(s)}\int_0^{\infty}\left(\sqrt[8]{e^x}\theta_2\left(0, e^{-x/2}\right)x^{s-1}\right)dx$$

where $\Gamma(s)$ is the gamma function and . $\theta_2(*,*)$ is Jacobi theta function of the second kind. But, he and the author do not know the way of analytic continuation of the function $P_2(s)$, yet.

## 5. Pythagorean triples on $M_3$

This research is by Y. Yabe (cf. pp167-180 in [2]). Let $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix}$. A triple $(A, B, C)$ is called the Pythagorean triple on $M_3$ when $(A, B, C)$ satisfies the two conditions (1) $A^2 + B^2 = C^2$, (2) numbers except for $a_2, b_2$ and $c_2$ are positive. For example when

$$A = \begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & 3 \\ 0 & 12 \end{pmatrix}, \quad C = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}$$

$(A, B, C)$ is a Pythagorean triples. In this example, putting $A' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, B' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}, C' = \begin{pmatrix} 5 & 3 \\ 0 & 5 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$A = LA'R,\ B = LB'R, C = LC'R \text{ and } A'^2 + B'^2 = C'^2.$$

From this case, Yabe defined a reducible $(A, B, C)$ as follows.

*Definition 1.* *$(A, B, C)$ is called reducible when $(A, B, C)$ satisfies the three conditions (1) there exist elements $A', B', C', L$, and $R$ of $M_3$ such that $A = LA'R, B = LB'R$ and $C = LC'R$, (2) $L$ and $R$ don't have invertible matrixes, (3) $A'^2 + B'^2 = C'^2$. Moreover, $(A, B, C)$ is called irreducible when $(A, B, C)$ is not reducible.*

Yabe thought if triples by numbers $(a_1, b_1, c_1)$ and $(a_3, b_3, c_3)$ were primitives then $(A, B, C)$ could be irreducible. However he found an example such that $(a_3, b_3, c_3)$ is not primitive and $(A, B, C)$ is irreducible. Namely,

$$A = \begin{pmatrix} 7 & 5 \\ 0 & 16 \end{pmatrix}, \quad B = \begin{pmatrix} 24 & 23 \\ 0 & 30 \end{pmatrix}, \quad C = \begin{pmatrix} 25 & 23 \\ 0 & 34 \end{pmatrix}.$$

Let $\wp$ be the set of Pythagorean triples on $M_3$. Yabe defined homomorphic map $f_h: M_3 \to M_3$ by $f_h(A) = \begin{pmatrix} a_1 & ha_2 \\ 0 & a_3 \end{pmatrix}$ for any integer $h$. Moreover, for any two triples $(A, B, C)$ and $(A', B', C')$ in $\wp$, he defined the relation $\sim$ on $\wp$ by $(A, B, C) \sim (A', B', C') \Leftrightarrow$ there exists a integer $h$ such that $f_h(A) = A', f_h(B) = B'$ and $f_h(C) = C'$. Then the relation $\sim$ becomes the equivalence relation. From this, we have a quotient set $\wp/\sim$. A triple $(A, B, C)$ with $\gcd(a_2, b_2, c_2) = 1$ is important because it be a representative of the equivalence class $[(A, B, C)]$. So Yabe defined $\wp_{h=1}$ to be the set of $(A, B, C)$ with $\gcd(a_2, b_2, c_2) = 1$. Let $A_h = f_h(A)$, $B_h = f_h(B)$ and $C_h = f_h(C)$.

*Definition 2.* *$(A, B, C)$ is reducible in $\wp_{h=1}$ when $(A, B, C)$ satisfies the three conditions (1) there exist an integer $h$ and elements $L$ and $R$ of $M_3$ such that, $A = LA_hR, B = LB_hR$ and $C = LC_hR$, (2) $L$ and $R$ don't have invertible matrixes, (3) $A_h^2 + B_h^2 = C_h^2$. Moreover, $(A, B, C)$ is irreducible in $\wp_{h=1}$ when $(A, B, C)$ is not reducible in $\wp_{h=1}$.*

Yabe proved the following theorem.

*Theorem 1. Let* $(A, B, C)$ *be a Pythagorean triple in* $\wp_{h=1}$*. Then* $(A, B, C)$ *is irreducible in* $\wp_{h=1}$*. if and only if* $(a_1, b_1, c_1)$ *and* $(a_3, b_3, c_3)$ *are primitive.*

## 6. ${}^{p}M_n$ perfect numbers

This research is by S. Kiriyama (cf. pp181-190 in [2]). Let $p$ be a prime number and ${}^{p}M_n$ be a $p$-Mersenne prime number which is defined by ${}^{p}M_n = 1 + p + \cdots + p^{n-1}$ is prime. First he found the following formula

$$p^{n-1} \times {}^{p}M_n = (p-1)\tau(p^{n-1} \times {}^{p}M_n) - (p-2)\,{}^{p}M_n$$

where $\tau(x)$ is the sum of proper positive divisors of $x$. From this, he had an idea of new perfect number.

*Definition 3. A natural number* $a$ *is called* ${}^{p}M_n$ *perfect number when* $a$ *satisfies the condition*

$$a = (p-1)\tau(a) - (p-2)\,{}^{p}M_n.$$

Kiriyama researched many ${}^{p}M_n$ perfect numbers by using a computer and he found a important number ${}^{p}K_{d,n}$ which was defined by $(p-1)\,{}^{p}M_d - (p-2)\,{}^{p}M_n$ where $d \geq n$. In particular, when $p = 2$, we have ${}^{2}K_{d,n} = {}^{2}M_d$ which is a Mersenne number. He proved that if ${}^{p}K_{d,n}$ is a prime number then $p^{n-1} \times {}^{p}K_{d,n}$ is a ${}^{p}M_n$ perfect number. So, Kiriyama made the following problem.

*Problem 1. Can every* ${}^{p}M_n$ *perfect number be represented by the from* $p^{n-1} \times {}^{p}K_{d,n}$*, where* ${}^{p}K_{d,n}$ *is a prime number?*

When $p = 2$ Problem 1 becomes the Theorem (Euclid-Euler). Kiriyama and the author think that to prove Problem 1 is difficult in the case $p \geq 3$. He proved the following theorem.

*Theorem 2. Assume that* $p \geq 3$ and $d = n$*. Then Problem 1 is correct.*

## 7. Conclusion

By experiencing mathematics research, we can discover some new mathematical concepts and a new mathematical world which we didn't know. Such discovery gives us great pleasure. Also in the three research works introduced in this paper, my students found new mathematical concepts, i.e., Pascal zeta functions, Pythagorean triples on $M_3$ and ${}^{p}M_n$ perfect numbers. By finding these, they were able to see new mathematics worlds. They got a lot of excitement by these discoveries and they wrote papers to express the mathematical worlds which they found. They obtained a lot of mathematical knowledge and many skills through these experiences. Their research papers were evaluated in Japan's most famous science research contests (cf. [3],[4]).

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# Three Wonderful Mathematics Research Papers by High School Students

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**概要：** 学生に数学の自由研究を経験させることは教育的に重要である。なぜならば，自主的に多くの知識とスキルを獲得できるからである。自由研究において学生達は，まず研究すべき問題を見つけなければならない。次に，彼らはその問題に対して，多くの適切なデータを集めなければならない。そしてデータからいくつかの法則を抽出しなければならない。その後，定理を作りそれを証明しなければならない。必要ならこれまで自分自身が知らなかった知識を学んだり，いくつかの論文を読まなければならない。そして最後に，行った研究をまとめ，論文を書かなければならない。本発表は，2015年と2017年に指導した著者の学生のオリジナルな数学研究を紹介するものである。そして取り上げる学生達の研究タイトルは，“パスカルゼータ関数”，“$M_3$上のピタゴラス数”そして“${}^{p}M_n$完全数”である。

本発表では，学生に数学の自由研究を指導する上で，未解決問題を鑑賞することを提案し，その方法を提示する。これは，自由研究のテーマの決め方がわからないほとんどの学生において有効な手段であると考えている。具体的には，未解決問題を鑑賞させ，その後，それにチャレンジさせるのではなく，鑑賞している未解決問題の重要性や面白さを議論させる方法である。このような議論を通して，彼らは“自分自身は何について取り組むべきか”ということを同時に考えるのである。その際，教師は，彼らのそれぞれのアイディアを汲み取って，研究しやすいテーマが立てられるようサポートをしなければならないし，さらに立てられたテーマが発展できるようアドバイスを与えなければなければならない。発表においては，アドバイスに関する２つの例を取り上げる。

さて，“パスカルゼータ関数”の研究は，パスカル三角形の２列目の自然数の列からリーマン・ゼータ関数$\zeta(s) = 1^s + 2^s + 3^s + 4^s + \cdots$をイメージし，その結果，３列目の三角数の列から新しいタイプのゼータ関数$P_3(s) = 1^s + 3^s + 6^s + 10^s + \cdots$を定義し研究した学生の自由研究の紹介である。“$M_3$上のピタゴラス数”の研究は，２次正方行列全体がなす環の部分環である$M_3 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \text{ は非負の整数} \right\}$に関する研究である。すなわち，$M_3$上のピタゴラス数$(A, B, C)$を，$M_3$の元$A, B, C$でかつ$A^2 + B^2 = C^2$を満たすもの，と定義した学生の研究である．そして彼は，“$M_3$上のピタゴラス数にも原始的なものを定義できるのだろうか？”という疑問を持ち，この疑問を解決するために研究をスタートさせた。“${}^{p}M_n$完全数”の研究は，通常の完全数が，2のべき乗の和で作られているメルセンヌ数$M_n$に関係することを独自に気付き，そこから，“素数$p$のべき乗の和で作られている新しいタイプのメルセンヌ数${}^{p}M_n$からも，完全数に類似した数というものがあるのではないか？”というアイディアから行われたものである。本発表では，以上の３つの自由研究から得られた結果を紹介する。これらの研究を行った学生達は，このような経験から多くの数学的知識とスキルを獲得しただけでなく，彼らの研究は日本の有名な科学コンテストで高く評価されている(cf. [3],[4])。

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# Grundy number of Divisor Nim

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**Abstract**: This is a study on "Divisor Nim" which is a kind of mathematical game of Two-players, Zero-sum, Perfection information Game. "Divisor Nim" is a game that imposes a limitation of "The number of stones that a player can be taken is about a divisor number (without 1 and n) of the current stone number n" on the general rule of Nim. In addition to the fun as a stone taking game, it is also a game with fun as number theory, because it is a game using divisor. We think that "Divisor Nim" is a good theme to train mathematical thinking abilities while playing games.

**Keywords**: Nim, Grundy number

## 1. Divisor Nim

Nim is a game where two players take stones alternately from mountains of n stones. Divisor Nim is a game which restricted Nim by the number of stones to be taken. In addition, Nim is a game "the player who could not get the stone at last loses". The number of stones that can be taken like Divisor Nim varies depending on the game, and the state of the mountain before the player takes the stone is called the state of the game.Below is a description of one of several rules and the end condition of the game.

Rule: The two players take stones alternately from the mountain consisting of N stones by the following rule. Each player takes only the number of true divisors (divisors without 1 and N) of "N" the number of stones at his turn.

The end condition of the game: The end condition of this game is that the player can't take a stone.

## 2. Grundy number

Grundy number means "Mapping from the state of the game to nonnegative integer" and is defined as follows.

(1) Grundy number of the end state is 0.

(2) When a state is not an end state, Grundy number of the state is the smallest nonnegative integer that is different from any of the Grundy number of the subsequent states.

"Minimal excluded number (=mex)" is introduced to formulate "minimum nonnegative integer different from any Grundy number" used in (2). "mex" is defined as mex (T) = min (N - T) for the set N of nonnegative integers and the true subset T of N. When P is each state in a game and N(P) is the total set of subsequent states of P, (2) can be written as g(P) = mex (g(N(P))). The Grundy number has the following properties.

**Theorem 1**

1) $g(k) = 0$ $\quad (k : odd, k \geqq 3)$

2) $g(2^{2\ell+1}) = 0$ $\quad (\ell \geqq 1)$

3) $g(2^{2\ell}) \neq 0$ $\quad (\ell \geqq 1)$

4) $g(2^{\ell}k) \neq 0$ $\quad (\ell \geqq 1, k : odd, k \geqq 3)$

**Theorem 2**

When $\ell \in \mathbb{N}$ and $N \geqq 161$, $g(N)$ is

1) $g(2^{2\ell}) = 2\ell - 1$

2) $g(2^{\ell}3) = \ell - 2$

3) $g(2^{\ell}) = \ell + 2$

4) $g(2^{\ell}k) = \ell + 1$ $\quad (k : odd, 7 \leqq k \leqq 15)$

5) $g(2^{\ell}k) = \ell$ $\quad (k : odd, k \geqq 17)$

Figure 1:Theorem 1,Theorem 2

Table 1:Grundy number from 1 to 320

| $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ | $N$ | $g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 21 | 0 | 41 | 0 | 61 | 0 | 81 | 0 | 101 | 0 | 121 | 0 | 141 | 0 | 161 | 0 | 181 | 0 | 201 | 0 | 221 | 0 | 241 | 0 | 261 | 0 | 281 | 0 | 301 | 0 |
| 2 | 0 | 22 | 1 | 42 | 1 | 62 | 1 | 82 | 1 | 102 | 1 | 122 | 1 | 142 | 1 | 162 | 1 | 182 | 1 | 202 | 1 | 222 | 1 | 242 | 1 | 262 | 1 | 282 | 1 | 302 | 1 |
| 3 | 0 | 23 | 0 | 43 | 0 | 63 | 0 | 83 | 0 | 103 | 0 | 123 | 0 | 143 | 0 | 163 | 0 | 183 | 0 | 203 | 0 | 223 | 0 | 243 | 0 | 263 | 0 | 283 | 0 | 303 | 0 |
| 4 | 1 | 24 | 4 | 44 | 2 | 64 | 5 | 84 | 2 | 104 | 3 | 124 | 2 | 144 | 5 | 164 | 2 | 184 | 3 | 204 | 2 | 224 | 6 | 244 | 2 | 264 | 3 | 284 | 2 | 304 | 4 |
| 5 | 0 | 25 | 0 | 45 | 0 | 65 | 0 | 85 | 0 | 105 | 0 | 125 | 0 | 145 | 0 | 165 | 0 | 185 | 0 | 205 | 0 | 225 | 0 | 245 | 0 | 265 | 0 | 285 | 0 | 305 | 0 |
| 6 | 2 | 26 | 1 | 46 | 1 | 66 | 1 | 86 | 1 | 106 | 1 | 126 | 1 | 146 | 1 | 166 | 1 | 186 | 1 | 206 | 1 | 226 | 1 | 246 | 1 | 266 | 1 | 286 | 1 | 306 | 1 |
| 7 | 0 | 27 | 0 | 47 | 0 | 67 | 0 | 87 | 0 | 107 | 0 | 127 | 0 | 147 | 0 | 167 | 0 | 187 | 0 | 207 | 0 | 227 | 0 | 247 | 0 | 267 | 0 | 287 | 0 | 307 | 0 |
| 8 | 0 | 28 | 2 | 48 | 3 | 68 | 2 | 88 | 4 | 108 | 2 | 128 | 0 | 148 | 2 | 168 | 3 | 188 | 2 | 208 | 5 | 228 | 2 | 248 | 3 | 268 | 2 | 288 | 6 | 308 | 2 |
| 9 | 0 | 29 | 0 | 49 | 0 | 69 | 0 | 89 | 0 | 109 | 0 | 129 | 0 | 149 | 0 | 169 | 0 | 189 | 0 | 209 | 0 | 229 | 0 | 249 | 0 | 269 | 0 | 289 | 0 | 309 | 0 |
| 10 | 1 | 30 | 1 | 50 | 1 | 70 | 1 | 90 | 1 | 110 | 1 | 130 | 1 | 150 | 1 | 170 | 1 | 190 | 1 | 210 | 1 | 230 | 1 | 250 | 1 | 270 | 1 | 290 | 1 | 310 | 1 |
| 11 | 0 | 31 | 0 | 51 | 0 | 71 | 0 | 91 | 0 | 111 | 0 | 131 | 0 | 151 | 0 | 171 | 0 | 191 | 0 | 211 | 0 | 231 | 0 | 251 | 0 | 271 | 0 | 291 | 0 | 311 | 0 |
| 12 | 3 | 32 | 0 | 52 | 2 | 72 | 4 | 92 | 2 | 112 | 5 | 132 | 2 | 152 | 3 | 172 | 2 | 192 | 4 | 212 | 2 | 232 | 3 | 252 | 2 | 272 | 4 | 292 | 2 | 312 | 3 |
| 13 | 0 | 33 | 0 | 53 | 0 | 73 | 0 | 93 | 0 | 113 | 0 | 133 | 0 | 153 | 0 | 173 | 0 | 193 | 0 | 213 | 0 | 233 | 0 | 253 | 0 | 273 | 0 | 293 | 0 | 313 | 0 |
| 14 | 1 | 34 | 1 | 54 | 1 | 74 | 1 | 94 | 1 | 114 | 1 | 134 | 1 | 154 | 1 | 174 | 1 | 194 | 1 | 214 | 1 | 234 | 1 | 254 | 1 | 274 | 1 | 294 | 1 | 314 | 1 |
| 15 | 0 | 35 | 0 | 55 | 0 | 75 | 0 | 95 | 0 | 115 | 0 | 135 | 0 | 155 | 0 | 175 | 0 | 195 | 0 | 215 | 0 | 235 | 0 | 255 | 0 | 275 | 0 | 295 | 0 | 315 | 0 |
| 16 | 2 | 36 | 2 | 56 | 4 | 76 | 2 | 96 | 6 | 116 | 2 | 136 | 3 | 156 | 2 | 176 | 5 | 196 | 2 | 216 | 3 | 236 | 2 | 256 | 7 | 276 | 2 | 296 | 3 | 316 | 2 |
| 17 | 0 | 37 | 0 | 57 | 0 | 77 | 0 | 97 | 0 | 117 | 0 | 137 | 0 | 157 | 0 | 177 | 0 | 197 | 0 | 217 | 0 | 237 | 0 | 257 | 0 | 277 | 0 | 297 | 0 | 317 | 0 |
| 18 | 1 | 38 | 1 | 58 | 1 | 78 | 1 | 98 | 1 | 118 | 1 | 138 | 1 | 158 | 1 | 178 | 1 | 198 | 1 | 218 | 1 | 238 | 1 | 258 | 1 | 278 | 1 | 298 | 1 | 318 | 1 |
| 19 | 0 | 39 | 0 | 59 | 0 | 79 | 0 | 99 | 0 | 119 | 0 | 139 | 0 | 159 | 0 | 179 | 0 | 199 | 0 | 219 | 0 | 239 | 0 | 259 | 0 | 279 | 0 | 299 | 0 | 319 | 0 |
| 20 | 3 | 40 | 4 | 60 | 2 | 80 | 3 | 100 | 2 | 120 | 4 | 140 | 2 | 160 | 6 | 180 | 2 | 200 | 3 | 220 | 2 | 240 | 5 | 260 | 2 | 280 | 3 | 300 | 2 | 320 | 8 |

Table 2: Grundy number when N is "an even number × an odd number"

| $\ell$ \ $k$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | … | 159 | 161 | … | 2n−1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | … | 0 | 0 | … | 0 |
| 1 | 0 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | … | 1 | 1 | … | 1 |
| 2 | 1 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | … | 2 | 2 | … | 2 |
| 3 | 0 | 4 | 4 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | … | 3 | 3 | … | 3 |
| 4 | 2 | 3 | 3 | 5 | 5 | 5 | 5 | 5 | 4 | 4 | … | 4 | 4 | … | 4 |
| 5 | 0 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 5 | 5 | … | 5 | 5 | … | 5 |
| 6 | 5 | 4 | 8 | 7 | 7 | 7 | 7 | 7 | 6 | 6 | … | 6 | 6 | … | 6 |
| 7 | 0 | 5 | 9 | 8 | 8 | 8 | 8 | 8 | 7 | 7 | … | 7 | 7 | … | 7 |
| 8 | 7 | 6 | 10 | 9 | 9 | 9 | 9 | 9 | 8 | 8 | … | 8 | 8 | … | 8 |
| 9 | 0 | 7 | 11 | 10 | 10 | 10 | 10 | 10 | 9 | 9 | … | 9 | 9 | … | 9 |
| 10 | 9 | 8 | 12 | 11 | 11 | 11 | 11 | 11 | 10 | 10 | … | 10 | 10 | … | 10 |
| 11 | 0 | 9 | 13 | 12 | 12 | 12 | 12 | 12 | 11 | 11 | … | 11 | 11 | … | 11 |
| … | … | … | … | … | … | … | … | … | … | … | … | … | … | … | … |

Property 1: Grundy number of state of victory goes to the one who makes the second move is zero. Otherwise Grundy number is not zero.

Property 2: If Grundy number of all subsequent states is not 0, Grundy number of that states is 0. If Grundy number in either subsequent states is 0, Grundy number in that state is not 0.

Property 3: Due to the nature of "mex", it is equivalent that Grundy number of a certain states is k and that there are subsequent states in which Grundy number is 0 to k - 1.

Using property 2, we gradually evaluate Grundy number each state of the game from the end state. Using property 1 we will analyze mathematical games.

## 3. Grundy number of Divisor Nim

The end state with this Divisor Nim is when N does not have a true divisor. That is g(0)=0, g(1)=0. Similarly, if p is a prime number, then g(p) = 0. Next we examined Grundy number when N is the composite number. In order to investigate Grundy number, we conducted numerical experiments with computers. Among the results, Grundy number when N is 320 or less is listed in Table 1. We predicted Theorem 1 from the results in Table 1 and prove it [2]. However, we have not yet figured out the Grundy number of Theorems 1-3) and 1-4). From Theorem 1, we thought that there was a relationship between a power of 2 and the Grundy number. We created a table with exponent of 2 in vertical axis and odd number in horizontal axis (Table 2). The shaded part of Table 2 is a case where N is less than 161, and theorem 2 does not hold. We predicted Theorem 2 from the results in Table 2 and prove it [3].The Grandy number can be calculated from the Theorem 2 by transforming it into the form of "power of 2 × odd number". We do not need to perform prime factorization of N to evaluate g (N), just calculate the odd number and prime factors 2 contained in N.

## 4. Application to education

We think that Nim will help you learn the nature of integers. The reason is that you can learn the nature of integers while playing in the game. By expressing integers by the number of stones, Nim is a good game that allows players to watch integers in their eyes, touch them with hands, and operate. Considering playing with Divisor Nim as an example, you can feel that the divisor of n is not "an integer dividing n", but "divide n stones equally into two or more groups". In addition, the player can feel that the number of stones "Anyone cannot be divided into two or more groups" is prime number. In addition to Divisor Nim, we are also studying "Prime Divisor Nim" and "Power of Prime Divisor Nim".These games are obtained by changing the number of stone to take from the divisor to the prime divisor, the power of the prime divisor. Their Grundy numbers are entirely different from Divisor Nim. Like this, changing the rules greatly changes Grundy number, and at the same time the gaming nature such as winning strategy changes. By changing to various rules, we think that we can also experience new knowledge of mathematics by rule.

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# 約数ニムのグランディ数

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## １．約数ニムのグランディ数の導出

一山崩し（ニム）とはn個の石からなる山から２人のプレイヤーが交互に石を取っていくゲームである．約数ニムはこの一山崩しに「各プレイヤーは自分の手番で石の個数nの真の約数個のみ取ってよい」という制限を加えたゲームである．また，「最後に石が取れなくなったプレイヤーの負け」としたゲームを正規系のゲームと呼び，ここでは約数ニムを正規系のゲームとして説明をする．約数ニムのように取れる石の個数はゲームによって異なり，プレイヤーが石を取る前の山の状態をゲームの局面と呼ぶ．グランディ数とは「ゲームの局面から非負整数への写像」のことであり，以下のように定める．

(1)ゲームの終了局面のグランディ数は0となる．

(2)ゲームの終了局面ではないゲームの局面のグランディ数は，後続の局面のグランディ数のどれとも異なる最小の非負整数となる．

グランディ数には「先手必敗局面のグランディ数は0である」「先手必勝局面のグランディ数は0ではない」という性質があり，この性質を用いて約数ニムの解析を行うことができる．

約数ニムのグランディ数を計算する際にコンピュータによる数値実験を行った．そのデータを元に定理を予想し，その証明も与えた[2][3].

## ２．山崩しから整数論を学ぶ

山崩しはゲームをしながら整数（石の個数）を目で見て，触って，操作ができる良いツールであると思われる．約数ニムを行うことを例に用いると，nの約数は「nを割り切る整数」ではなく，「n個の石を２つ以上のグループに平等に分ける」として体感できるだろう．

取れる石の個数をルールによって変えると，グランディ数は大きく変化するので，また別のゲーム性を楽しむことができる．約数ニムのような「取れる個数を約数のみとする」というものから様々なルールに変えることで，新たな数学の知識も得られるだろう．

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# Enhancement of Input Type for Math e-Learning System STACK

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**Abstract**: Mathematics e-learning system have been common in recent years and we can assess students' answers containing mathematical expressions true or false automatically by using computer algebra system. However, there are some challenges when using the system and one of them is inconvenience in inputting mathematical expressions as answers. In order to overcome the problem, some math-input interfaces have been developed. We propose 'FlickMath' using which students can easily input mathematical expressions by flick operations. The input interface is especially useful in the case of using mobile devices. Furthermore, we developed note-submitting function as an interface through which students can submit answers with related notes on calculations.

**Keywords**: Mathematics e-Learning, math-input interface

## 1. Introduction

In recent years, Mathematics e-learning system have been attracting interests for online test related to scientific subjects. The system can assess students answer submitted as mathematical expressions, which is contrast to the multiple-choice type questions that are usually common for online test. The mathematical expression as answers are evaluated true of false by using a computer algebra system. For example, when students answer $3x^2 - \frac{2x}{(x^2+1)^2}$ to the question of differentiation $\frac{d}{dx}\left(x^3 + \frac{1}{x^2+1}\right)$, they have to enter the expression `3*x^2-2*x/(x^2+1)^2` in the answer column. However, if students enter the expression in which numbers and symbols are mixed, typing errors would be easily caused. When students use smartphone for the online test, it would be more difficult to enter the expression because switching keyboard between alphabet and numbers/symbols are required. In order to increase input efficiency in STACK, MathTOUCH[1] and interface utilizing MathDox formula editor[2] have been proposed. However, they are assumed to be used mainly in PC. In order to reduce the complexity in entering mathematical expressions using mobile devices, we develop flickable math input interface[3], which is expected to increase the opportunities of drill practice using online test by using mobile devices as smartphones. Furthermore, it is sometimes not adequate to solely evaluate answers and it would be better to receive note in which some calculations are written. We develop a note-submitting function for STACK[4] as a report-type plug-in for the quiz module of Moodle and an input-type plug-in for STACK.

## 2. Flickable Math Input Interface: FlickMath

Nakamura et al. implemented MathDox formula editor as a new input interface for STACK[2] as an input type. Based on the MathDox input type, FlickMath was developed especially for using STACK on mobile devices[3] with which students can input mathematical expressions by the flick operation. The flick operation is carried out by placing a finger on the prepared keyboard, shifting the finger vertically or laterally, and subsequently releasing it. The left panel of Figure 1 shows an example of input of mathematical expression by using FlickMath. We estimated an input efficiency of FlickMath compared to normal keyboard on mobile devices and it was remarkable that the number of key touches is overall decreased. However, student cannot enter many letters such as '$v$' and '$t$' with the early numerical FlickMath keyboard, which is inconvenient to carry out online test for physics. The letters '$v$' and '$t$' are usually used for variables of velocity and time in physics. Then we implemented the flick operation using a traditional keyboard. The right panel of Figure 1 shows an example using FlikMath extended to a traditional full keyboard. Additionally, the numerical keyboard and traditional keyboard automatically appear in smartphones and tablets respectively by default depending on the screen size. It is possible to switch between these keyboards. MathDox input process is used for PC.

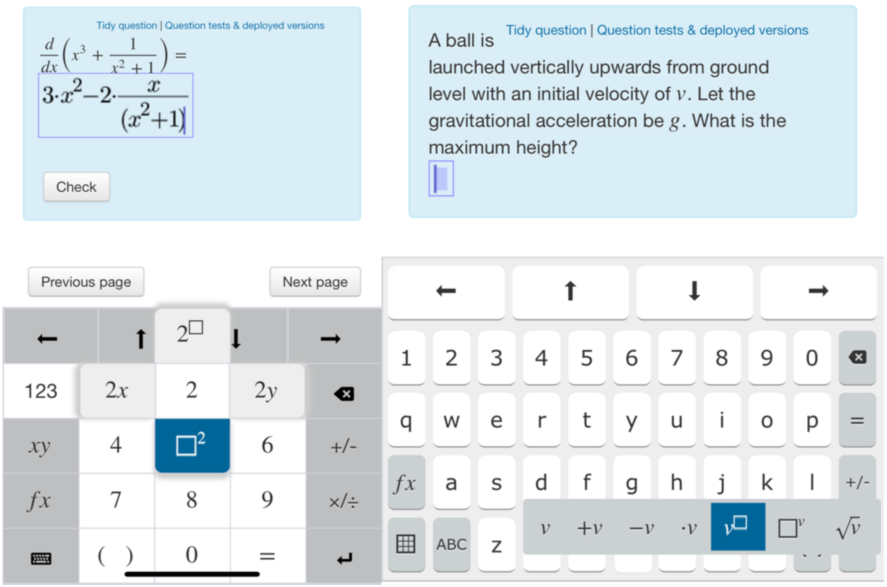


*Figure 1. Examples of FlickMath keyboard, the numerical keyboard (left) and extended full keyboard (right).*

## 2. Note-Submitting Function

Let us consider the question of solving an ordinary nonhomogeneous differential equation (ODE) $\frac{dy}{dx} - 2y = e^x$. When students submit $y(x) = Ce^{2x} - e^x$, STACK can evaluate the answer as correct. STACK is not aware of how the students solved the question. However, there are two typical methods of solving the ODE, the method of variation of parameters and the method of undetermined coefficients and it is important for teachers to know the method that students adopt to solve the ODE. Then we develop a new STACK function with which students can submit an answer together with calculation notes and teachers can view students' notes together with their submitted answers. The function was realized by developing an input-type plug-in for STACK and a report-type plugin for the quiz module of Moodle. Students can submit notes by uploading images of the corresponding notes or by hand-writing the notes directly onto the device. Teacher can also insert comments on the notes and return them to the students for later learning.

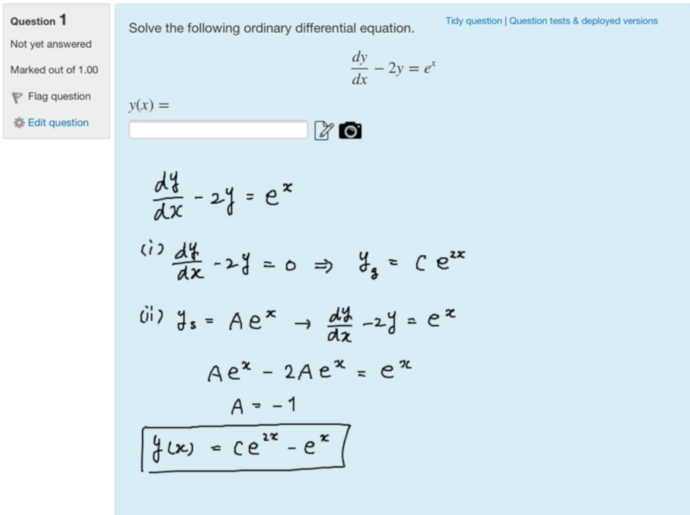

*Figure 2. Submitted notes by writing on the tablet devices.*

## 2. Conclusion

We developed a math input interface with flick operation for taking online mathematics test using STACK, which is considered to be useful for increasing drill practice by using mobile devices. Furthermore, we realized the note-submitting functions for STACK and teachers can view students' answer together with notes in which students write calculation process. Teacher can also insert comments on the notes.

We comment that the note-submitting function can also be implemented to the other question type like multiple-choice question type.

## Acknowledgements

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# 数学ｅラーニングシステムSTACKの入力タイプの拡張

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ｅラーニングシステムのオンラインテストを用いて，学生の理解度を測るために従来多く採用されてきたのは，多肢選択式の問題タイプであった。複数の解答候補の中から正答を選択する場合，当て推量で解答したものが正答となる，いわゆる偽正答など，学生の理解度を正しく測ることができないという潜在的な問題が含まれている。数学を始めとする自然学の分野で，多肢選択式のオンラインテストにより学生の理解度を測ろうとする場合，当て推量による偽正答の問題の他に，計算問題などで，単純に正答か誤答かを判断するだけでなく，計算問題などで部分点を与えたい場合などの対応困難性も指摘しなければならない。このような背景から，計算問題の解答として学生は数式を入力し，それを評価することのできる，数式自動採点システム（数学ｅラーニングシステム）が近年注目されている。数式の自動採点には数式処理システム(CAS)が用いられるが，大阪府立大学ではwebMathematicaをベースとして，先駆的にMATH ON WEBを開発した。その他には，CASとしてMapleを利用したMaple T.A.，Maximaを利用したSTACKの活用が国内で広まりつつある。

これら数学ｅラーニングシステムを活用するにあたり，問題点として指摘されることの一つは，解答として数式を入力する時の煩雑さである。例えば，$\frac{d}{dx}\left(x^3+\frac{1}{x^2+1}\right)$という微分を計算する問題の場合，$3x^2-\frac{2x}{(x^2+1)^2}$ が正答であるが，それを入力するには，各システムが採用するCASの書式に従う必要があり，STACKでは `3*x^2-2*x/(x^2+1)^2` と入力することが求められる。一次元的な表記のため分数をイメージすることが困難であったり，括弧が多いと入力ミスも誘発されたりするなど，数学の能力を測る以前に，学生が躓く可能性がある。このような問題を開発するために，MathTOUCH[1]やMathDox[2]などの数式入力インターフェースが開発されてきた。一方，時間や場所を問わず学習することを可能にするために，スマトーフォンなどのモバイルデバイスを用いたオンラインテスト環境も多く提供されていきているが，数学ｅラーニングの場合，上述の数式入力の問題はより深刻になる。数字，アルファベット，記号が混在した数式では，モバイルデバイス上でキーボードの切り替えが頻繁に要求されるからである。

このような背景から，我々はモバイルデバイス上での数学ｅラーニングを促進するための環境として，日本語入力では広く用いられているフリック操作を活用した数式入力インターフェースを開発した[3]。これにより，一つの数式を入力するためのキータッチ数を大幅に削減することができた。当初はテンキータイプのもののみであったが，任意の文字入力を可能にするために，伝統的なフルキーボードでもフリック操作により数式入力可能なキーボードに拡張されている。

また，自然科学分野のテストでは，結果だけでなくそれを得るための計算，思考過程を評価することも重要である。しかし，数学ｅラーニングに限らずオンラインテストでは結果としての解答の正誤評価のみが中心であった。そこで，解答とともに，そこに到達するまでの計算，思考過程を記したノート提出機能を開発した[4]。教師はそのノートを確認し，必要に応じて添削したり，コメントを加えたりすることができ，学生の自主学習につながると期待される。この機能はSTACKの入力タイププラグイン，Moodleの小テストモジュールのレポートタイププラグインとして開発されたものであるが，STACKに限らず他の問題タイプにも応用できると考えられる。

# Extension of Output Function in MathTOUCH for the Production of Mathematical Materials

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**Abstract**: Our efforts to lighten the burden imposed when digitally entering mathematical formulae led us to propose a new math input editor, named MathTOUCH. The editor uses predictive conversion to convert obscure linear strings presented in a colloquial style into suitable mathematical expressions. This paper describes an application in which MathTOUCH is implemented in an e-assessment system for mathematics. An investigation of the effectiveness of our system, carried out by conducting an online learning test, revealed a high level of user satisfaction. However, a thinking (offline) process is also important for advanced mathematics learning. In this study, we developed an output function in Microsoft Word format for MathTOUCH to reduce the burden imposed on users when producing digital mathematics materials.

**Keywords**: Mathematical material editor, Math input method, Intelligent user interface

## 1. Introduction

One of the problems associated with mathematics learning based on a digital tool was caused by the troublesomeness of inputting mathematical formulae. Especially, it was cumbersome for novice students to have to learn system operations to be able to enter a formula digitally as an answer in the e-assessment system as opposed to the traditional way of mathematics learning.

## 2. Intelligent math input interface: MathTOUCH

To address the shortcoming in Section 1, we developed a math input editor, named MathTOUCH [1], which accepts linear strings entered in a colloquial style. This method displays a list of candidates for the desired mathematical expression in a WYSIWYG editor. After all the elements are interactively chosen, the desired expression is formed and outputted in the desired format. This mathematical input process is illustrated in *Figure 1*.

The rules for a linear string for a mathematical expression are as follows: the key letter (or word) for an objective mathematical symbol consists of the ASCII code(s) corresponding to the initial or clipped form (such as the LaTeX form); a single key often supports many mathematical symbols.

For example, when a user wants to input $\frac{\alpha^2+1}{2}$, the linear string is denoted by "a2+1/2," where "a" represents the "alpha" symbol. It is unnecessary to include the power sign (i.e., the caret character (^)) and the parentheses for the numerator, because they are not printed. Other representative cases are shown in *Table 1*. For example, the linear string for $e^{\pi x}$ is denoted by "epx." However, the linear string of the expressions $e_p x$, $e^{px}$, and $e^{\pi}x$ are also denoted by "epx." Hence, there are some ambiguities when representing mathematical expressions as linear strings using these rules.

To address this shortcoming on such obscure notation, we proposed a predictive algorithm [2] to convert a linear string into the most suitable mathematical expression using a hypothesis function after perceptron machine learning by using a data set consisting of 4000 mathematical formulae. An experimental evaluation using test data containing general mathematical formulae achieved a prediction accuracy of 85% for the top ten ranking.

MathTOUCH enables users to input almost any mathematical expression dealt with in the widespread categories of mathematics from junior high school level to university level without learning a complex language such as LaTeX. Additionally, its performance in terms of input repeatedly improves by virtue of the real-time machine-learning function for suitable prediction. The available output formats are LaTeX, MathML, PNG, JPEG, EPS, Maxima, Maple, and Mathematica.

*Table 1. Examples of MathTOUCH rules*

| Category | Linear strings | Formulae |
|---|---|---|
| variable | a | $a$ or $\alpha$ |
| polynomial | x2-3x+2 | $x^2 - 3x + 2$ |
| fraction | 4/3 | $\frac{4}{3}$ |
| square root | root5 | $\sqrt{5}$ |
| trigonometric | sin2t | $\sin^2 \theta$ |
| logarithm | log10x | $\log_{10} x$ |
| exponent | epx | $e^{\pi x}$ |
| summation | sumk=1nk2 | $\sum_{k=1}^{n} k^2$ |
| integral | intabf(x)dx | $\int_a^b f(x)\, dx$ |

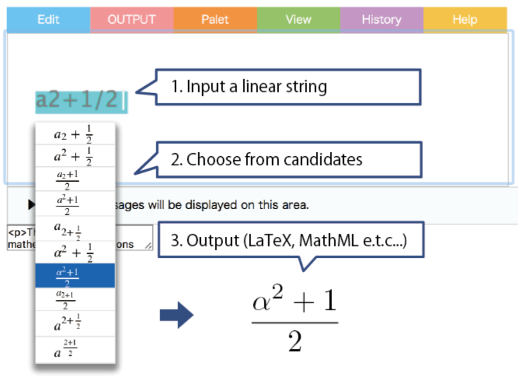


*Figure 1. Input Process on MathTOUCH*

## Matrix Input Function

MathTOUCH allows users to input an expression in the form of an $m \times n$ matrix of which each element is any mathematical formula as shown in *Figure 2*; i.e., vector algebra and a simultaneous equation. First, users activate the start/end of matrix mode by inputting the '#'-character key and specify the desired size by adding rows and/or columns by using [shift]+ '→' key and/or [shift]+ '↓' key combinations, respectively. Then, the $m \times n$ text-fields are placed in $m$ rows and $n$ columns as the linear strings of matrix elements. *Figure 3* shows a screenshot of the matrix input functionality.

## Application to e-assessment system for Mathematics

MathTOUCH has been developed using JavaScript (HTML5). Therefore, developers are able to

$$\begin{pmatrix} 1 & -2 \\ \frac{3}{2} & x^2 \end{pmatrix}$$

$$\begin{cases} x + y = 1 \\ 2x - 3y = 5 \end{cases}$$

*Figure 2. Representative examples of Matrix*

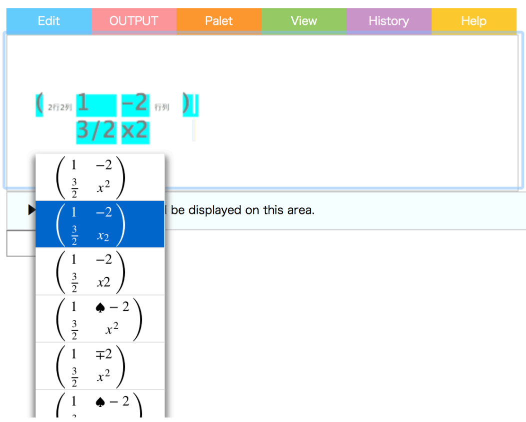


*Figure 3. Matrix inputting by MathTOUCH*

incorporate MathTOUCH into their own web applications. For example, we implemented MathTOUCH in the e-assessment system STACK [3] for Mathematics on Moodle to enable students to enter a mathematical formula directly as their answer in response to a Mathematics question. *Figure 4* shows a screenshot of the interface on STACK. In our previous study, we conducted an eight-week learning experiment involving simple mathematical calculation work as in *Figure 5* to evaluate the efficacy of MathTOUCH. The result showed that students were able to practice using MathTOUCH at the same learning rate as with the current interface on STACK. Furthermore, the results of the questionnaire revealed a higher level of satisfaction regarding Memorability.

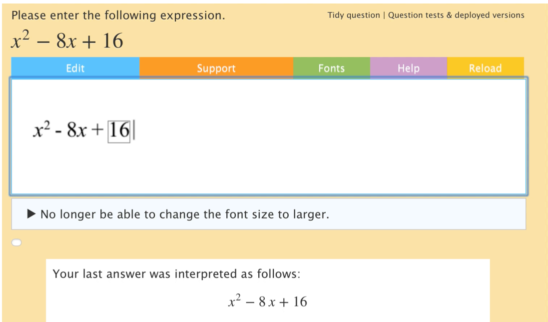


*Figure 4. MathTOUCH on STACK*

$$\sqrt{20} \times 2\sqrt{2} \div \sqrt{5}$$

$$\sqrt{50} - 4\sqrt{2} + \frac{6}{\sqrt{2}}$$

*Figure 5. Example of Math work*

## 3. Extension of output function in MathTOUCH

However, achieving mathematics learning only by using drill type online learning such as mentioned above is insufficient. More specifically, the thinking process is important for advanced mathematics learning as required for a calculation and/or proof. In this study, we have extended the output function in MathTOUCH to enable users to use the linear string format of Microsoft Word for mathematical expressions. Therefore, the use of this function is expected to lighten the burden imposed upon non-mathematics students to digitally record mathematics notes or reports, namely their thinking process, by using MathTOUCH and Microsoft Word. Furthermore, this function would help teachers to create original mathematics teaching materials.

## 4. Summary

In this paper, we presented an output function for Microsoft Word format in MathTOUCH to reduce the burden imposed on users when producing digital mathematics materials. The most important avenues for future research are to evaluate the efficiency of this function in MathTOUCH by conducting a student subject test of the usability.

## Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 17K00501.

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# MathTOUCHにおける数学文書作成のための出力機能の拡張

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## １．はじめに

デジタルツールを使った数学学習における課題の一つは、数式のデジタル入力の煩わしさにある。特に、数学初学者である学生が数式による答えを直接入力する数学eラーニング等の学習では、本来の学習とは別の、システムに合わせた操作や式表現の文法を覚える必要があり、負担である。

## ２．曖昧入力による数式構築UI：MathTOUCH

我々は第１章で述べた問題を改善するために、利用者が所望する数式を普段読むような曖昧入力だけで、機械が候補を予測して、教科書と同じ表記で構築できる、新しい数式入力方式を提案した。さらにこの方式を実装した数式入力インタフェースMathTOUCHを開発した[1]。

MathTOUCHは理工系大学数学で扱われているほとんどの数式を入力することが可能で、数式要素を含む行列入力にも対応している。また、MathTOUCHはJavaScriptで開発しており、様々な数学ソフトウェアに組み込んで利用することを想定している。Webアプリケーション公開版では、HTML5が動作するほとんどのマシンで利用可能である。構築された数式は、所望の形式で出力する機能があり、OSを介して任意のアプリケーションに貼り付け可能である。現在対応している出力形式は、数学論文やWebページのためのマークアップ言語形式（LaTeX, MathML），画像形式（JPEG, PNG, EPS），数式処理や数式自動採点システムで使われている数式処理システムのコマンド形式（Maxima, Maple, Mathematica）である。

先行研究では、数式自動採点システムSTACK on Moodleの数式入力インタフェースにMathTOUCHを実装し、数学ドリル問題の学習実験を８週にわたり行ったところ、従来入力方式と変わらずスムースに学習が進み、利用者の満足度を有意に高めることができた[2]。

## ３．数学文書作成のための機能拡張

しかし、数学学習において、このようなオンライン学習だけでは不十分で、途中計算過程や証明問題など考えるオフライン学習も重要であることは確かである。本研究ではその思考過程の記録をデジタル文書に記録しやすくするため、MathTOUCHにおける数式出力機能を拡張し、Microsoft Wordの数式線形形式に対応した。これにより理工系の学生自身が数学レポートやノートなどをデジタルに記録する負担を軽減できるものと期待される。さらには、教員が独自の数学教材を作成する際にも役立つものと考える。今後、このWord出力機能を実際に学生に使ってもらい、被験者実験評価を行う予定である。

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# MeLQS: Mathematics e-Learning Questions Specification

## A common base for sharing questions among different systems

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**Abstract**: Computer-aided online assessment systems for mathematics, such as STACK, Maple T. A., and MATH ON WEB, which are usually referred to as mathematics e-learning systems, are used for university mathematics education. Those systems can assess student answers submitted as mathematical expressions as opposed to only as multiple-choice options as usually used for online quizzes. When we use mathematics e-learning systems, it is important to develop content, but this need usually causes a heavy workload. If teachers could share content or resources, their burdens would be reduced. In order to make it possible to share content even among different systems, it is preferable to aggregate content within a common base. We suggest Mathematics e-Learning Question Specification (MeLQS) as this common base. MeLQS is constructed with two specifications: concept design and implementation specification. MeLQS would determine effective ways to carry out mathematics e-learning.

**Keywords**: Mathematics e-Learning, Question Sharing

## 1. Introduction

In recent years, information and communication technology infrastructure has been improved in schools, and e-learning has become increasingly popular. One of the most important features of e learning is computer-aided assessment of students' answers. The most common question type in an online assessment is the multiple-choice question (MCQ) type, in which the potential answers are provided for students to select their answer. However, the MCQ type may not be sufficient to evaluate students' understanding levels. For example, students can sometimes guess correctly even if they do not understand the answer. Therefore, it is better to adopt a question type in which students provide answers as mathematical expressions, especially in scientific subjects. Mathematical expressions as answers are evaluated as true or false by a computer algebra system (CAS). For example, for the problem of differentiation $\frac{d}{dx}(x+1)^2$, the correct answer is $2(x+1)$, but some students may provide $2x + 2$, or others may provide $2 + 2x$, and so on. These answers are all mathematically equivalent answers that and should be evaluated to be as correct answer. In recent years, this kind of mathematics e-learning system have has been becoming gaining popularity, and with mainly STACK[1], Maple T.A.[2], and MATH ON WEB[3,4], are used in Japan.

When we conduct online mathematics tests using mathematics e-learning systems, it is important to prepare contents or questions, and it is convenient if that content can be shared among systems. Tens of thousands of questions for Maple T.A. are available in the “Maple T.A. Cloud” [5], which is a worldwide content-sharing system covering a variety of subjects, including calculus, algebra, differential equations, physics, chemistry, and so on. “Mathbank” [6] is open to the public as a Moodle system for sharing questions for STACK. Users can download STACK questions in XML format from Mathbank and import the file to their servers for subsequent use. The question-sharing systems Maple T.A. Cloud and Mathbank are designed for the specific systems Maple T.A. and STACK, respectively. There is a conversion tool from Maple T.A. to STACK [7], but the conversion is not always perfect. Therefore, in order to promote mathematics e-learning further, it would be preferable to share questions among different mathematics e-learning systems.

## 2. The Necessity of a Common Base for Sharing Questions

We have reviewed the above-mentioned two question-sharing systems, and it is undoubtedly preferable to share questions among different systems to accumulate content. In order to realize this, it is necessary to have a common base for sharing questions. If we develop content with a common base, it could be easier to rebuild questions for each mathematics e-learning system. Given this background, we suggest Mathematics e-Learning Question Specification (MeLQS), which we believe helps determine effective ways to carry out mathematics e-learning.

## 3. Mathematics e-Learning Questions Specification: MeLQS

In order to build a common base for sharing questions, we verified the structures of the question data in STACK and MATH ON WEB’s Web-based Assessment System of Mathematics (WASM). Our aim is to share the questions in the universal format MeLQS, which is expected to be easy to export to any format of mathematics e-learning systems, including MATH ON WEB, STACK, and Maple T.A. After preliminary analysis of the structures of the questions on MATH ON WEB and STACK, we found it appropriate to categorize the parts of each question as follows: question text and procedure to create it; definition of answer column and answer type; procedureto the authoring tool to evaluate student answer and give feedback. MeLQS is constructed with two specifications: “concept design” and “implementation specification.” These specifications handle metadata of questions: question name, subject, intention behind a question, etc.

Concept design is a specification of questions that describes their concepts. Questions are created according to the concept design, which is described by mathematical statements rather than programming statements so that all mathematics teachers can understand the concept. Figure 1 shows an example of concept design that describes a question on linear algebra. In order to create a concept design including metadata, we implemented an authoring tool as a Moodle plug-in. We are planning to implement MathTOUCH as a math input interface so that all teachers can edit mathematical expressions more intuitively.

Implementation specification for questions is designed for those who have experience authoring questions for online tests based on the suggested concept design. In the implementation specification, details of settings of questions defined as dependencies on each math e-learning system are eliminated. For example, input of a mathematical expression should not be dependent on a specific CAS syntax.

Questions based on the implementation specification can be exported as a suitable format for any mathematics e-learning system.

We plan to provide MeLQS as a cloud service with which users can design and author questions. Authored questions can be exported for implementation to various systems.

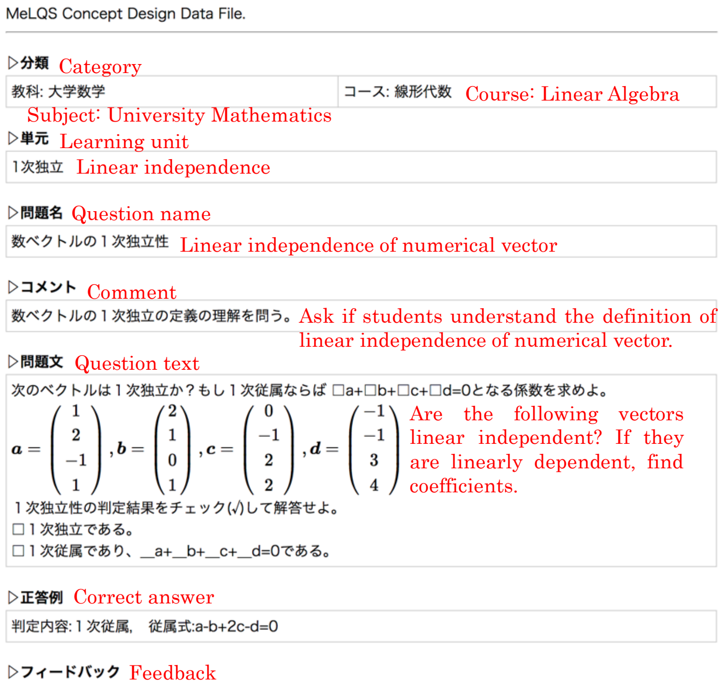


*Figure 1. Example of concept design*

## 4. Conclusion

We started a four-year project in 2016 and aim to share questions among different mathematics e learning systems, especially MATH ON WEB, STACK, and Maple T.A. at present, based on the universal format MeLQS. MeLQS is constructed with two specifications: concept design and implementation specification. We plan to provide MeLQS as a cloud service that enables users to create questions for different systems, and we believe that heavy use of the service will promote mathematics e-learning.

## Acknowledgements


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# MeLQS: Mathematics e-Learning Questions Specification

# 異種数学 e ラーニングシステムにおける問題共有のための標準仕様


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|---|---|---|---|
| 大阪府立大学 | 大阪府立大学 | 三玄舎 | 名古屋大学 |
| 福井哲夫 | 白井詩沙香 | 加藤克也 | 谷口哲也 |
| 武庫川女子大学 | 武庫川女子大学 | サイバネットシステム | 日本大学 |


従来のオンラインテストで一般的に用いられていた多肢選択式では十分に測ることのできない計算力，思考力を問うために，数式で提示された解答の正誤評価を可能にする，数式自動採点システム（以下，数学 e ラーニングシステム）が注目されている。日本で主に利用されているシステムは，MATH ON WEB[1,2]，STACK[3]，Maple T.A.[4]であるが，これらのシステムを効果的に運用するために最も重要なことは，多くの良質なオンラインテストの問題を蓄積し利用することである。効率的かつ効果的に問題を蓄積するためには，問題を共有することが有効であるが，Maple T.A.の問題を共有するためにはMaple T.A. Cloud[5]が，STACKの問題を共有するためにはMathbank[6]が提供されている。しかし，各システムで共有される問題はMaple T.A.かSTACKのいずれかのみであり，システム相互で共有されてはいない。今後，数学 e ラーニングを推進していくためには，システムの違いという壁を超えて問題を共有することにより，良問を有効に活用することが望まれる。

我々は，そのような背景から，数学 e ラーニングコンテンツの標準化による，異種システム間連携を実現するための標準仕様MeLQS (Mathematics e-Learning Question Specification)を提案し，その仕様データから各システムの問題を作成する仕組みの構築を行っている。

MeLQSでは，数学オンラインテストの問題データの構造を，問題文および問題文作成ルーチン，解答欄などの解答スタイルの定義，解答判定ルーチンとフィードバックに分類し，問題のメタデータとして，科目・単元などの分類，出題意図，問題名などを定義している。そして，MeLQSの大きな特徴は，問題がどのような意図でどのようにデザインされたかを記述する「問題仕様（Concept Design）」と，システムに実装する際に必要な情報を記述する「実装仕様（Implementation Specification）」の二段階式を採用した点である。

問題仕様では，上記のデータ構造の内容を，数式処理などのプログラムの記法ではなく，内容の把握が容易な数学的記述で記載することを想定している。したがって，オンラインテストだけでなく，紙などのテストを設計する際にも有用となる。我々は，この問題仕様書を作成するためにMoodleのプラグインとして仕様書作成ツールを開発した。このツールを用いることにより，ステップ・バイ・ステップで問題仕様書を編集することができる。

実装仕様は2017年12月現在，規格策定段階であるが，数学オンラインテストの問題作成の経験があるユーザが，問題仕様に基づいて実装仕様書を作成することを想定している。実装仕様に基づいて作成された問題は，各システムの問題形式にエクスポートして利用することを可能にすることが目標である。

MeLQSは数学 e ラーニングコンテンツの異種システム間連携を実現するための標準仕様として提案したものであるが，今後実装仕様の規格策定とその作成ツールの開発を行っていく予定である。また，問題仕様書，実装仕様書を作成する際，システム毎に異なる数式処理システムの書式の違いを吸収することのできる数式入力インターフェースを，MathTOUCH[8]を利用して実装する予定である。

International Workshop on Mathematics Education
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# Trends and considerations on the college level mathematics needs at one-to-one mathematics tutoring centers for adults

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## 1. Introduction

The Wakara Corporation has been offering private mathematics tutoring sessions for adults for approximately seven years. Since its establishment, we have tutored adults of all ages and responded to their various mathematics needs, including those related to college-level mathematics. Currently, we have three tutoring centers in Tokyo and one in Osaka; in contrast to tutoring centers intended for children, our company does not offer mathematics education to assist in school entrance exams; instead, we explore the possibilities of mathematics for people and our society and promote activities and events to introduce cutting-edge mathematics to society. In this work, we analyzed the need for "college level mathematics" for adults who come to our one-to-one tutoring center.

## 2. Background

We analyzed the trends among customers who attend our mathematics tutoring center and take "college level mathematics." In this analysis, we focused on "areas of mathematics requested" and "purpose of taking the course." To obtain the most up-to-date data, we targeted 200 students who are currently registered (as of October 31, 2017) and learning "college level mathematics."

Since the target group consists of the customers taught by our company, it is easy to assume that there is social demand, or that the customer has a personal need to learn mathematics, and that the customer's financial situation is such that they will be able to pay the tutoring fee. It is important to note that the study is not targeting any group composed of those who can study by themselves. There are a very limited number of other tutoring centers for adults available, and thus it is thought that our company holds the largest market share.

In this analysis, we defined "college level mathematics" as a curriculum that is not taught in elementary-to-high school mathematics and calculus, and this is categorized into the following areas: Mathematics (higher than college level), Statistics, Economy/Finance/Financial Engineering, Physics (higher than college level), and others.

## 3. Results

First, Figure 1 shows the analysis results for the "areas of mathematics requested," Second, Figure 2 shows the analysis results for the "purpose for taking the course." Third, Figure 3 shows which areas are required for their "work" because most people are taking the tutoring to assist in their jobs.

## 4. Three customer case studies

A. Mr. Sato (male) (pseudonym)

He joined a financial consulting company as a new employee in April. There are four other new hires, and they all studied statistics in college. He graduated from the literature department and did not study statistics. He is concerned that he lacks statistical knowledge compared to the other four employees. His job requires statistical knowledge, so he is studying little by little, but even in high school he only studied freshman-level basic mathematics, so it is difficult for him to understand the mathematical terms. He is coming to the end of the company's training period and needs to be ready for the actual job; however, without the statistical knowledge he needs, it will be difficult to do a good job, which is why he requested help. The tutoring began with an explanation of the words he will be using at his job, including terms such as logistic regression, decision tree, odds, and risk ratio.

B. Mr. Suzuki (pseudonym) (male in his 30s)

He is performing wave analysis at his current job, but is planning to be assigned to the development of medical ultrasonic treatment devices at his next job. At both jobs, he is asked to have higher mathematical and physics knowledge, and therefore needs to learn both urgently. Goal 1: To understand Fourier series/transform, coordinate transformation (matrix, determinant). Goal 2: To understand college level physics (vibration and wave motion, ultrasonics, directionality). He has started studying Fourier transform by himself using the internet, but has not started learning college level physics. Started tutoring sessions on rotation matrix and determinant, as well as the Fourier series.

C. Mr. Kato (pseudonym) in his 20s

He is working for a marketing research company on business analysis. He feels that he has some level of statistical knowledge, but conducts analysis without knowing the logistics behind the same. As for his future career development, he would like to learn new analytical methods, including how to understand the mathematical backgrounds of the same. Also, a part of his job requires R, and in order to finish the work, he needs to be able to use R within 3 months. The required methods are multiple regression analysis, factor analysis, principal component analysis, and clusters.

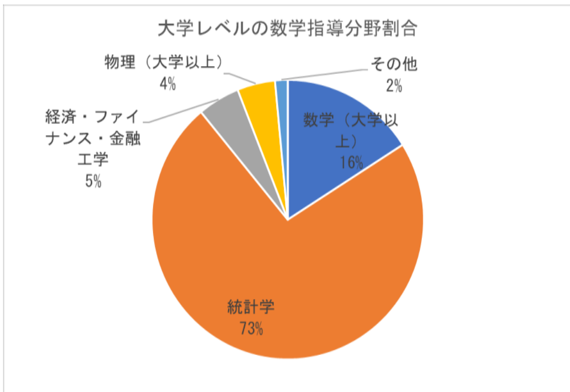


*Fig. 1. Tutoring areas*

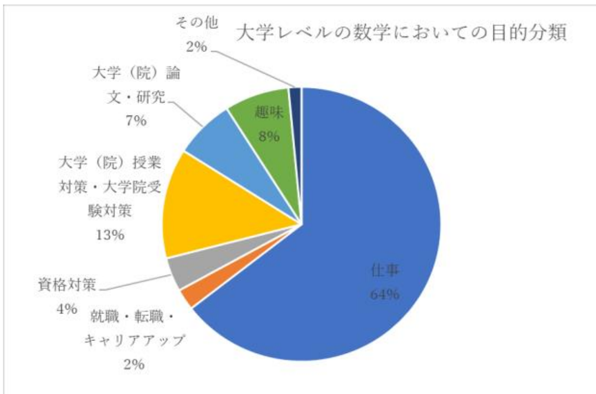


*Fig. 2. Purpose for seeking tutoring*

Fig 1 legend

大学レベルの数学指導分野割合: Ratio of college level mathematics tutoring areas　, 数学（大学以上）: Mathematics (higher than college level)　, 統計学: Statistics　, 経済・ファイナンス・金融工学: Economy/Finance/Financial Engineering　, 物理（大学以上）: /Physics (higher than college level)　, その他：Other

Fig 2 legend

大学レベルの数学においての目的分類：Purposes for taking college level mathematics　, 趣味:

Hobby ，仕事：Work ，就職・転職・キャリアアップ：Job hunting/business skills development ，資格対策：Exam preparation ，大学（院）授業対策・大学院受験対策：College (under and graduate level) lecture support/ Graduate school entrance exam preparation ，大学（院）論文・研究: College (under and graduate level) report and research，その他：Other

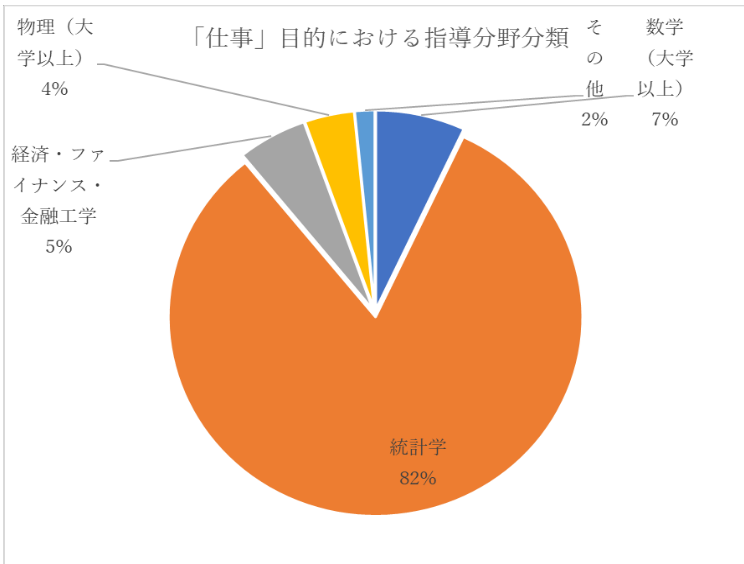


*Fig.3 Tutoring categories for "work" purposes*

Fig 3 legend
「仕事」目的においての目的分類：Areas of mathematics taken for work purposes ，統計学: Statistics ，経済・ファイナンス・金融工学: Economy/Finance/Financial Engineering ，物理（大学以上）: /Physics (higher than college level) ，数学（大学以上）: Mathematics (higher than college level) ，その他：Other

## 5. Discussion

AI-related topics have been making the news in recent years, and the need for "statistics" is much higher than other areas. There is little need for more than one in six people to learn the linear algebra taught by college mathematics departments. To meet our customer's needs, we have been offering "mathematics tutoring," including statistics, and our program has been expanding since 2013, with most customers asking to learn statistics. In fact, three out of four customers request statistics in terms of higher mathematical education. Two out of three people utilize tutoring for "work" purposes, and 80% of the people who seek tutoring for work purposes request statistics. The percentage of people taking tutoring sessions as a hobby is 8%. However, most likely, there are those who would like to take tutoring, but cannot do so because of the high cost of the one-to-one tutoring. Thus, this indicates there is a higher level of hidden demand and more potential need.

These observations raise questions of continuing current mathematical education curriculums offered at colleges. Isn't statistics the most required area in society, rather than mathematics? It is true that this could be a temporary boom, so it requires further and careful consideration, but successful methods, including the machine learning method and deep learning, have created a need to learn statistics in companies. The author hopes this report will bring forth a wave of new mathematics education curriculum development in colleges.

# 大人向け数学個別指導塾に対する大学数学のニーズの傾向と考察

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## １．概要

弊社和から株式会社は社会人向け数学教室として、設立から約7年が立ち、多くの社会人に数学を指導し、現在では東京3教室、大阪1教室と展開、幅広い年齢層に対し多様な数学（大学数学を含む）ニーズに応えてきた。子供向け学習塾と大きく違い、受験のための数学教育を行う組織ではなく、人や社会に対する数学の在り方を探求し、最先端の数学を社会に発信するイベント活動等も行っている。今回は、個別授業教室に通って頂いている社会人の「大学レベルの数学」ニーズを具体的に分析する。

## ２．分析にあたって

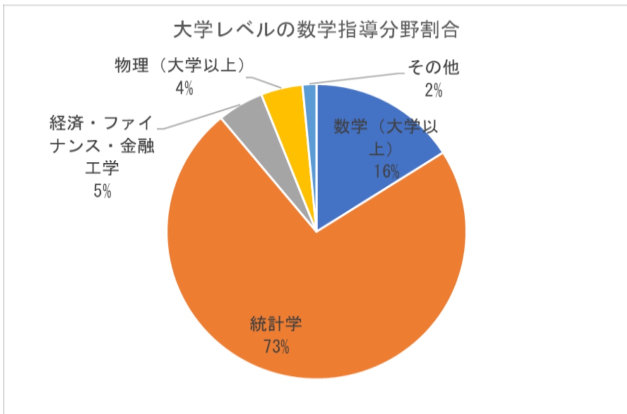
弊社社会人向け数学教室で受講するお客様の中で「大学レベルの数学」（定義：小学校から高校までの算数及び数学で学ばないカリキュラムとする）について特性分析を行った。データの最新性を保つため、2017年10月31日現在弊社に入会のお客様、かつ、「大学レベルの数学」として受講している最新200名をピックアップし、分析した。

データの特徴として弊社の個別指導に通うお客様であることから、社会的要請、本人の強いニーズがあり、かつ、お金を払えるという経済状態の方が容易に想像できる。自分で独学することのできる層の方は対象としていないことに注意が必要である。他に大人向け数学教室の数は少なく、弊社が対象市場において最大手であると思われる。

## ３．結果

右図のような結果となった。

図１．指導分野割合




## ３．考察

昨今、「人工知能」関連の話題がニュースとなる時代背景もあり、「統計学」へのニーズが圧倒的に高い。大学の数学科で学ぶような線形代数などの分野もニーズがあるにはあるにはあるが、6人中1人にも満たない。弊社のお客さまニーズとして、統計学も教えている「数学教室」として展開しているにもかかわらず、2013年頃から統計学のお客様が増え続け、高度な数学分野としては実に全体の4人中3人近くが希望する。ここから考察するに、本当に今の大学の数学科（他学科の数学科目）で学ぶカリキュラムが現状でいいのか、という疑念を抱かざるを得ない。本レポートがこれからの大学数学教育新設計へのきっかけとなることを期待したい。

# A Concluding Remark

## 結び

# Comments in the End

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**Abstract**: Our great appreciation goes to nice lectures and comments by the guests from abroad, and to this wonderful workshop realized by the organizers, who are promoting intensively the study of the improvement of math education in universities.
Here are two comments. The first one is that we should learn more from the mathematics education at schools, in particular at elementary schools, where Japan has a long and effective tradition of teacher education. The second is that the importance of statistics shows the necessity of the total change of mathematics education itself including the university education.
**Keywords**: Learning from Elementary School, Statistics Education in Math Education

## 0. Acknowledgement

First of all we would like to express our hearty thanks to all speakers, who made this workshop meaningful and valuable. Our appreciation goes in particular to the invited talks and many useful comments during the workshop by two guests from abroad, from whom we could learn new theories and practices done outside Japan. Our hearty thanks go also to the organizers of this wonderful workshop.

Here we remark that this kind of study to improve mathematics education in university was once tried in 90's by the Mathematical Society of Japan. The commentator was involved in that movement, which, however, could not succeed fully because of a big change of systems of universities. Also at that time main object was the improvement of the first-year course of math education in scientific or economic faculties. He is very glad that Prof. Mizumache and Prof. Kawazoe has begun this study again intensively and even more widely, including "all" students in universities.

## 1. First Comment

Our first comment is that the mathematics education of universities has much to be learned from that of elementary schools. This is what the commentator himself has learned in the recent years in involving with the study of mathematics education and teaching future math teachers. Japan has a long tradition of teacher education at elementary and lower-high schools. It is introduced even internationally (e.g. [1]). The commentator believes that university mathematics also should learn from elementary mathematics, because all the roots of university mathematics lie in elementary mathematics (discussed partially in [2]. Also see [3]).

## 2. Second Comment

Our second comment is that the workshop has shown that the education of statistics is of great importance. This is a new point of the 21st century, when we compare it with the movement before in 90's (at that time the treatment of computer science played a similar role). Because of the explosive development of computer, the study of mathematics itself is changing for adapting to data science such as the theory of big data. This fact is generally admitted, but the commentator would like to emphasize that mathematics education is responsible to the education of statistics to give its solid basis. Therefore, the education of mathematics (not only university mathematics but also school math) should change radically to include stochastic or probabilistic approaches. The most difficult issue is the logic including uncertainty. Moreover, one should be conscious of intimate relations with usual mathematics. For example, note that the relation of the cumulative frequency distribution and histograms is nothing but the discrete version of division quadrature.

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# 結びのコメント

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## 0．謝辞

まず初めに，この研究会を価値ある有意義なものとして下さったすべての講演者に心から感謝したい。就中国外からの２名の招待者は，素晴らしい講演に加え，研究会中常に適切なコメントを与えて下さった。それらにより新しい理論や国外での実践を学ぶことができた。またこの素晴らしい研究会を実現して下さった組織委員の方々にも心から感謝したい。

一つ述べておきたい。大学での数学教育を改善しようとする試みは日本数学会によって1990年代に行われた。小職はそれに関わったが，当時の大学制度の大きな変革（大綱化・大学院化等）のために中途で挫折した。また当時の主な目標は自然科学系あるいは経済系学部での初年次数学教育の改善であった。水町氏や川添氏達がこのテーマを再び取り上げて強力に研究を推進され，しかも「教養教育」を含む，より広汎な学生を対象とされていることに心からの敬意を表する。

## コメント１．大学数学教育は算数教育に学べ.

最初に，大学数学教育は小学校の数学教育（算数教育）からもっと多くを学ぶべきであると言いたい。これは小職自身がこの十年教育学部で教員養成に関わった中で痛感したことである。日本の数学義務教育（小中学校）の教員は高い質を保ち，しかも校内で彼等を育ててゆく効果的な伝統を持っている（授業研究）。これは今や国際的にも知られている（[1]）。小職は大学数学教育ももっと算数教育から学ぶべきであると信じている。なぜならすべての大学数学は初等数学の基盤の上に立っているからである。これについては部分的に[2]で論じた。また[3]をも参照。

## コメント２．数学教育は統計教育を視野に根本的に変わる必要がある

第２のコメントは，研究会が統計教育の重要性を示したことである。これは21世紀の新たなポイントである（90年代には計算機科学が似た役割を果たした）。前世紀後半からのコンピュータの爆発的進歩は，「ビッグデータ」理論などデータサイエンスに適合すべく，数学研究自身を変えつつある。これは周知の事実であるが，小職は統計教育に確かな基礎を与える数学教育の役割を強調したい。それ故に数学教育が（大学教育だけでなく学校教育も）確率論的方法論を含む形で根本的に変わらねばならない。最も難しい課題は不確実性を含む論理である。しかし更に従来の数学との密接な関係にも注意を払う必要がある。例えば累積度数分布とヒストグラムとの関係は区分求積法の離散バージョンに他ならない。

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